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On the dynamics of WKB wave functions whose phase are weak KAM solutions of H-J equation

Thierry Paul* Lorenzo Zanelli†

September 27, 2013

Abstract

In the framework of toroidal Pseudodifferential operators on the flat torus $\mathbb{T}^n := (\mathbb{R}/2\pi\mathbb{Z})^n$ we begin by proving the closure under composition for the class of Weyl operators $\mathrm{Op}^w_\hbar(b)$ with simbols $b \in S^m(\mathbb{T}^n \times \mathbb{R}^n)$. Subsequently, we consider $\mathrm{Op}^w_\hbar(H)$ when $H = \frac{1}{2}|\eta|^2 + V(x)$ where $V \in C^\infty(\mathbb{T}^n;\mathbb{R})$ and we exhibit the toroidal version of the equation for the Wigner transform of the solution of the Schrödinger equation. Moreover, we prove the convergence (in a weak sense) of the Wigner transform of the solution of the Schrödinger equation to the solution of the Liouville equation on $\mathbb{T}^n \times \mathbb{R}^n$ written in the measure sense. These results are applied to the study of some WKB type wave functions in the Sobolev space $H^1(\mathbb{T}^n;\mathbb{C})$ with phase functions in the class of Lipschitz continuous weak KAM solutions (of positive and negative type) of the Hamilton-Jacobi equation $\frac{1}{2}|P + \nabla_x v_\pm(P,x)|^2 + V(x) = \bar{H}(P)$ for $P \in \ell\mathbb{Z}^n$ with $\ell > 0$, and to the study of the backward and forward time propagation of the related Wigner measures supported on the graph of $P + \nabla_x v_\pm$.

Keywords: Toroidal Pseudodifferential operators, Wigner measures, Hamilton-Jacobi equation.

*CNRS and CMLS, Ecole Polytechnique (Palaiseau), thierry.paul@math.polytechnique.fr

†CMLS, Ecole Polytechnique (Palaiseau), lorenzo.zanelli@ens.fr

Contents

1 Introduction

In this paper we study WKB type wave functions on flat torus $\mathbb{T}^n := (\mathbb{R}/2\pi\mathbb{Z})^n$, namely functions of the form

$$\psi(x) = a(x)e^{iS(x)/\hbar},\ x \in \mathbb{T}^n,\ n \geq 1 \tag{1.1}$$ def1

where $a = a_{\hbar,P}$ is a family of functions in $L^2(\mathbb{T}^n;\mathbb{R})$ and $S(x) = P \cdot x + v(x),\ P \in \ell\mathbb{Z}^n,\ \ell > 0$, $\hbar^{-1} \in \ell^{-1}\mathbb{N}$, the phase $v(x) = v(P,x)$ is a Lipschitz continuous weak KAM solution of the stationary Hamilton-Jacobi equation

$$H(x, P + \nabla_x v(P,x)) = \bar{H}(P), \tag{1.2}$$ def-eff0-intr

for Hamitonian $H(x,\xi) := \frac{1}{2}|\xi|^2 + V(x)$, $V \in C^\infty(\mathbb{T}^n)$, see Section 2.2.1 for precise definitions.

It is well known that in the case where v is a regular function, the wave function ψ is, under general conditions on the family $a = a_{\hbar,P}$, a Lagrangian distribution associated to the Lagrangian manifold $\Lambda_P := \{(x,\eta) \in \mathbb{T}^n \times \mathbb{R}^n,\ \eta = P + \nabla_x v(P,x)\}$. Therefore it has an associated monokinetic Wigner measure of the form

$$dw(x,\eta) = \delta(\eta - (P + \nabla_x v(P,x)))|a_0(x)|^2 dx. \tag{1.3}$$ mono1

Moreover it remains of the same type under propagation through the Schrödinger equation whose quantum Hamiltonian is the quantization of the function $H(x,\xi)$ (see Section 2.1 for details on the toroidal quantization) leadding to a Wigner measure

$$dw^t(x,\eta) = \delta(\eta - (P + \nabla_x v(P,x)))|a_0^t(x)|^2 dx \tag{1.4}$$ wig2

where $|a_0(x)|^2$ satisfies a transport equation in such a way that dw^t is the pushforward of dw by the Hamiltonian flow of H.

The goal of this paper is to show what remains of this construction in the case where v is a solution of (1.2) with only a Lipschitz continuity property, a regularity far for being used in the framework of standard microlocal analysis.

Note that propagation of monokinetic Wigner measures with low regularity momentum profiles and application to the classical limit of propagation of WKB type wave functions have been recently studied in [4]. The regularity assumption in [4] is much stronger than ours, but at the contrary the construction in [4] works for any profile with a given regularity as we need our phase function to be a solution of the Hamilton-Jacobi equation. Therefore the two papers are complementary.

The precise definition of our WKB state, especially of the amplitude in (1.1), is given in Section 4.2, Definition 4.2 where a family of examples are given in the remark 4.3 following the definition.

Note that WKB states on the torus with phase functions issued form weak KAM theory have been used in [9], [10] where it has been studied L^2-energy quasimode estimates. In [23] a class of WKB states on the torus with regularized phase function have been defined in such a way the associated Wigner measures are coinciding with the Legendre transform of the so-called Mather measures.

In the present paper we will work with the true solution of Hamilton-Jacobi equation for the phase and will use a kind of regularization for the amplitude, as no canonical function choice is offered for the latter out weak KAM theory.

Our first main result concerns the Wigner measure w, as defined in Section 2.1.3, Definition 2.6, associated to our family of WKB states. It claims, Theorem 4.8, that w is as expected monokinetic in the sense that it has the form

$$dw(x,\eta) = \delta(\eta - (P + \nabla_x v(P,x)))dm_P(x) \tag{1.5}$$

where the limit in the measure sense $dm_P(x) = \lim_{\hbar\to 0} |a_{\hbar,P}(x)|^2 dx$ exists by Definition 4.2. In fact, we also assume that $dm_P \ll \pi_\star(dw_P) =: d\sigma_P$ where dw_P is the Legendre transform of a Mather P-minimal measure (see Section 2.2.2). This setting implies that any measure $dw(x,\eta)$ as in (1.5) is asbolutely continuous to dw_P itself, as shown in Lemma 4.7. We also underline that $d\sigma_P$ solves the continuity equation

$$0 = \int_{\mathbb{T}^n} \nabla_x f(x) \cdot (P + \nabla_x v(P,x))\, d\sigma_P(x) \quad \forall f \in C^\infty(\mathbb{T}^n), \tag{1.6}$$

and this can be interpreted as the result of an asymptotic free current density condition for the wave functions ψ of type (1.1), as we show in Proposition 4.10. We recall that in the usual construction of WKB wave functions (working with integrability or almost-integrability assumptions on H) the determination of the amplitude function $a(x)$ is related to the solution of the continuity equation (1.6) written in the strong sense for the function $\sigma(x) = a^2(x)$, namely $\mathrm{div}_x[(P+\nabla_x v(P,x))\sigma(x)] = 0$.

The above assumption on dm_P together with the monokinetic form of dw_P on the graph of a weak KAM solution of the Hamilton-Jacobi equation allow to study very much easily the time propagation of such measures, which remains of monokinetic type. This is in fact our second main result, which deals with the classical limit of the Wigner transform of the evolved WKB state. It is contained within Theorem 5.1 and Proposition 5.3 where the propagation,

$$dw^t(x,\eta) = \delta(\eta - (P + \nabla_x v(P,x)))g(t,P,x)dm_P(x) \tag{1.7}$$

both forward and backward (they are different in our situation) in time is exhibited.

The paper is organized as follows: Section 2 is devoted to some preliminaries concerning the Weyl quantization on the torus (2.1) and the weak KAM theory (2.2). Section 3 concerns the dynamics of the Wigner transform on the torus and Section 4 the classical limit of the Wigner transform, including the Section 4.2 where the monokinetic property of the Wigner function of our WKB state is established. Its propagtion is studied in the final Section 5.

2 Preliminaries

preli

2.1 The Weyl quantization on the torus

Weyl

2.1.1 Settings

Let us consider the flat torus $\mathbb{T}^n := (\mathbb{R}/2\pi\mathbb{Z})^n$. The class of symbols $b \in S^m_{\rho,\delta}(\mathbb{T}^n \times \mathbb{R}^n)$, $m \in \mathbb{R}$, $0 \leq \delta,\ \rho \leq 1$, consisting of those functions in $C^\infty(\mathbb{T}^n \times \mathbb{R}^n; \mathbb{R})$ which are 2π-periodic in x (that is, in each variable x_j, $1 \leq j \leq n$) and for which for all $\alpha, \beta \in \mathbb{Z}^n_+$ there exists $C_{\alpha\beta} > 0$ such that $\forall$ $(x,\eta) \in \mathbb{T}^n \times \mathbb{R}^n$

$$|\partial_x^\beta \partial_\eta^\alpha b(x,\eta)| \leq C_{\alpha\beta m} \langle \eta \rangle^{m-\rho|\alpha|+\delta|\beta|} \tag{2.8}$$ symb00

where $\langle \eta \rangle := (1+|\eta|^2)^{1/2}$. In particular, the set $S^m_{1,0}(\mathbb{T}^n \times \mathbb{R}^n)$ is denoted by $S^m(\mathbb{T}^n \times \mathbb{R}^n)$.
The toroidal Pseudodifferential Operator associated to $b \in S^m(\mathbb{T}^n \times \mathbb{R}^n)$ reads

$$b(X,D)\psi(x) := (2\pi)^{-n} \sum_{\kappa \in \mathbb{Z}^n} \int_{\mathbb{T}^n} e^{i\langle x-y,\kappa\rangle} b(y,\kappa)\psi(y)dy, \quad \psi \in C^\infty(\mathbb{T}^n;\mathbb{C}), \tag{2.9}$$

see [25]. In particular, we have a map $b(X,D) : C^\infty(\mathbb{T}^n) \longrightarrow \mathcal{D}'(\mathbb{T}^n)$. We recall that $u \in \mathcal{D}'(\mathbb{T}^n)$ are the linear maps $u : C^\infty(\mathbb{T}^n) \longrightarrow \mathbb{C}$ such that $\exists\, C > 0$ and $k \in \mathbb{N}$, for which $|u(\phi)| \leq C \sum_{|\alpha| \leq k} \|\partial_x^\alpha \phi\|_\infty$ $\forall \phi \in C^\infty(\mathbb{T}^n)$, see for example Definition 2.1.1 of [17]. Given a symbol $b \in S^m(\mathbb{T}^n \times \mathbb{R}^n)$, the (toroidal) Weyl quantization reads

$$\mathrm{Op}^w_\hbar(b)\psi(x) := (2\pi)^{-n} \sum_{\kappa \in \mathbb{Z}^n} \int_{\mathbb{T}^n} e^{i\langle x-y,\kappa\rangle} b(y,\hbar\kappa/2)\psi(2y-x)dy, \quad \psi \in C^\infty(\mathbb{T}^n). \tag{2.10}$$ weyl

Hence, it follows

$$\mathrm{Op}^w_\hbar(b)\psi(x) = (b(X, \frac{\hbar}{2}D) \circ T_x\,\psi)(x) \tag{2.11}$$ eq-O

where $T_x : C^\infty(\mathbb{T}^n) \to C^\infty(\mathbb{T}^n)$ defined as $(T_x\psi)(y) := \psi(2y-x)$ is linear, invertible and L^2-norm preserving. Starting from quantization in (2.10), we now introduce the Wigner transform $W_\hbar\psi$ by

$$W_\hbar\psi(x,\eta) := (2\pi)^{-n} \int_{\mathbb{T}^n} e^{2\frac{i}{\hbar}\langle z,\eta\rangle} \psi(x-z)\psi^\star(x+z)dz, \quad \eta \in \frac{\hbar}{2}\mathbb{Z}^n, \tag{2.12}$$ WignerT

which is well defined also for $\psi \in L^2(\mathbb{T}^n)$. For $b \in S^m(\mathbb{T}^n \times \mathbb{R}^n)$ the Wigner distribution reads

$$\langle \psi, \mathrm{Op}^w_\hbar(b)\psi \rangle = \sum_{\eta \in \frac{\hbar}{2}\mathbb{Z}^n} \int_{\mathbb{T}^n} b(x,\eta) W_\hbar\psi(x,\eta)dx, \quad \psi \in C^\infty(\mathbb{T}^n). \tag{2.13}$$ med4

For $b \in S^0(\mathbb{T}^n \times \mathbb{R}^n)$ and $\psi \in L^2(\mathbb{T}^n)$, the mean value $\langle \psi, \mathrm{Op}^w_\hbar(b)\psi \rangle_{L^2(\mathbb{T}^n)}$ is well defined thanks to the L^2 - boundedness estimate of $\mathrm{Op}^w_\hbar(b)$, see Theorem 2.3.

Remark 2.1. *Before to recall the notion of toroidal symbols and toroidal amplitudes, we need first to remind the notion of partial difference operator $\triangle$. Given $f : \mathbb{Z}^n_\kappa \longrightarrow \mathbb{C}$, it is defined the*

$$\triangle_{\kappa_j} f(\kappa) := f(\kappa + e_j) - f(\kappa) \tag{2.14}$$

where $e_j \in \mathbb{N}^n$, $(e_j)_j = 1$ and $(e_j)_i = 1$ if $i \neq j$. The composition provide $\triangle^\alpha_\kappa f(\kappa) := \triangle^{\alpha_1}_{\kappa_1} f(\kappa) \ldots \triangle^{\alpha_n}_{\kappa_n} f(\kappa)$ for any $\alpha \in \mathbb{N}^n_0$. We recall now that toroidal simbols $\widetilde{b} \in S^m_{\rho,\delta}(\mathbb{T}^n \times \mathbb{Z}^n)$, $m \in \mathbb{R}$, $0 \leq \delta,\ \rho \leq 1$, are those functions which are smooth in x for all $\kappa \in \mathbb{Z}^n$, 2π-periodic in x and for which for all $\alpha, \beta \in \mathbb{Z}^n_+$ there exists $C_{\alpha\beta m} > 0$ such that $\forall$ $(x,\kappa) \in \mathbb{T}^n \times \mathbb{Z}^n$

$$|\partial_x^\beta \triangle^\alpha_\kappa\ \widetilde{b}(x,\kappa)| \leq C_{\alpha\beta m} \langle \kappa \rangle^{m-\rho|\alpha|+\delta|\beta|} \tag{2.15}$$ symb-D

where $\langle\kappa\rangle := (1+|\kappa|^2)^{1/2}$. *As usually,* $S^m(\mathbb{T}^n\times\mathbb{Z}^n)$ *stands for* $S^m_{1,0}(\mathbb{T}^n\times\mathbb{Z}^n)$. *In the same way, it is defined the set of toroidal amplitudes* $S^m_{\rho,\delta}(\mathbb{T}^n\times\mathbb{T}^n\times\mathbb{Z}^n)$.
The link between this class of simbols and the euclidean ones $S^m_{\rho,\delta}(\mathbb{T}^n\times\mathbb{R}^n)$ *is shown within Theorem 5.2 in [25]. More precisely, for any* $\widetilde{b}\in S^m_{\rho,\delta}(\mathbb{T}^n\times\mathbb{Z}^n)$ *there exists* $b\in S^m_{\rho,\delta}(\mathbb{T}^n\times\mathbb{R}^n)$ *such that* $\widetilde{b}=b|_{\mathbb{T}^n\times\mathbb{Z}^n}$, *and conversely for any* b *there exists* $\widetilde{b}$ *such that this restriction holds true. Moreover, the extended simbol is unique modulo a function in* $S^{-\infty}(\mathbb{T}^n\times\mathbb{R}^n)$.

Remark 2.2. *In [16] it is considered the phase space Fourier representation,*

$$b(x,\eta)=F(\widehat{b}):=(2\pi)^{-n}\int_{\mathbb{R}^n}\sum_{q\in\mathbb{Z}^n}\widehat{b}(q,p)e^{i(\langle p,\eta\rangle+\langle q,x\rangle)}dp,\qquad (q,p)\in\mathbb{Z}^n\times\mathbb{R}^n,\quad (x,\eta)\in\mathbb{T}^n\times\mathbb{R}^n, \tag{2.16}$$

pf-T

(in the sense of distributions) and the operator $U_\hbar(q,p)\psi(x):=e^{i(q\cdot x+\hbar p\cdot q/2)}\psi(x+\hbar p)$ *which is well defined on* $L^2(\mathbb{T}^n)$ *for any fixed* $(q,p)\in\mathbb{Z}^n\times\mathbb{R}^n$. *In this framework, the Weyl quantization of a simbol* $b\in S^m(\mathbb{T}^n\times\mathbb{R}^n)$ *is given by*

$$\mathrm{Op}^w_\hbar(b)\psi(x):=(2\pi)^{-n}\int_{\mathbb{R}^n}\sum_{q\in\mathbb{Z}^n}\widehat{b}(q,p)U_\hbar(q,p)\psi(x)dp. \tag{2.17}$$

Q2

Consequently, the corresponding Wigner transform and Wigner distribution are

$$\widehat{W}_\hbar\psi(q,p) := \langle\psi, U_\hbar(q,p)\psi\rangle_{L^2}. \tag{2.18}$$

$$\langle\psi,\mathrm{Op}^w_\hbar(b)\psi\rangle := \int_{\mathbb{R}^n}\sum_{q\in\mathbb{Z}^n}\widehat{b}(q,p)\widehat{W}_\hbar\psi(q,p)dp. \tag{2.19}$$

In fact, the Weyl quantizations as in (2.10) and (2.17) are coinciding (see Proposition 2.3 in [23]).

2.1.2 Composition and Boundedness for Weyl operators

In the following we recall a result on $L^2(\mathbb{T}^n)$-boundedness for a class of operators involved in our paper.

Th-Bound0 **Theorem 2.3** (see [16]). *Let* $\mathrm{Op}^w_\hbar(b)$ *as in (2.17) with* $b\in S^0_{0,0}(\mathbb{T}^n\times\mathbb{R}^n)$. *Let* $N=n/2+1$ *when* n *is even,* $N=(n+1)/2+1$ *when* n *is odd. Then, for* $\psi\in C^\infty(\mathbb{T}^n)$

$$\|\mathrm{Op}^w_\hbar(b)\psi\|_{L^2(\mathbb{T}^n)}\le\frac{2^{n+1}}{n+2}\,\frac{\pi^{(3n-1)/2}}{\Gamma((n+1)/2)}\sum_{|\alpha|\le 2N}\|\partial^\alpha_x b\|_{L^\infty(\mathbb{T}^n\times\mathbb{R}^n)}\ \|\psi\|_{L^2(\mathbb{T}^n)}. \tag{2.20}$$

By using standard arguments (such as Hahn-Banach Theorem, see for example [24]) the above class of operators can be extended as bounded linear operators on $L^2(\mathbb{T}^n)$.This is the toroidal counterpart of the well known Calderon-Vaillancourt Theorem for Pdo on $\mathbb{R}^n$ (see for example [21]).

We devote now our attention to the composition of these toroidal operators (see also [16], for a similar result involving a smaller class of simbols).

Th-comp **Theorem 2.4.** *Let* $\ell,m\in\mathbb{R}$, $a\in S^\ell(\mathbb{T}^n\times\mathbb{R}^n)$ *and* $b\in S^m(\mathbb{T}^n\times\mathbb{R}^n)$. *Then,*

$$\mathrm{Op}^w_\hbar(a)\circ\mathrm{Op}^w_\hbar(b)=\mathrm{Op}^w_\hbar(a\sharp b) \tag{2.21}$$

comp0

where $a\sharp b=a\cdot b+O(\hbar)$ *in* $S^{\ell+m}(\mathbb{T}^n\times\mathbb{R}^n)$. *Moreover,*

$$[\mathrm{Op}^w_\hbar(a),\mathrm{Op}^w_\hbar(b)]=\mathrm{Op}^w_\hbar(a\sharp b-b\sharp a) \tag{2.22}$$

comm1

where the Moyal bracket reads $\{a,b\}_{\mathrm{M}}:=a\sharp b-b\sharp a=-i\hbar\{a,b\}+O(\hbar^2)$ *in* $S^{\ell+m-1}(\mathbb{T}^n\times\mathbb{R}^n)$.

Proof. To begin, we observe that $T_\omega \psi(y) := \psi(2y-\omega)$ can be written as

$$T_\omega \psi(y) = (2\pi)^{-n} \sum_{\kappa \in \mathbb{Z}^n} \int_{\mathbb{T}^n} e^{i\langle (2y-\omega)-z,\kappa\rangle} \psi(z) dz, \quad \forall\ \psi \in C^\infty(\mathbb{T}^n;\mathbb{C}). \tag{2.23}$$

By Theorem 8.4 in [25], it follows

$$\mathrm{Op}^w_\hbar(b)\psi(x) = (b(X, \frac{\hbar}{2}D) \circ T_{\omega=x}\psi)(x) = (2\pi)^{-n} \sum_{\kappa \in \mathbb{Z}^n} \int_{\mathbb{T}^n} e^{i\langle x-z,\kappa\rangle} c(\hbar, x, z, \kappa)\psi(z) dz \tag{2.24}$$

with *toroidal amplitude* $c(\hbar,\cdot) \in C^\infty(\mathbb{T}^n_x \times \mathbb{T}^n_z \times \mathbb{Z}^n_\kappa)$ such that $|\partial_x^\alpha \partial_z^\gamma c(\hbar,x,z,\kappa)| \leq C_{\alpha\gamma}\langle\kappa\rangle^{\ell+m}$. In particular, $c = b(z, \frac{\hbar}{2}\kappa) + O(\hbar)$ in $S^m(\mathbb{T}^n \times \mathbb{T}^n \times \mathbb{Z}^n)$. Now, apply Theorem 4.2 in [25], so that there exists a unique toroidal simbol $\sigma(\hbar,\cdot) \in S^m(\mathbb{T}^n \times \mathbb{Z}^n)$ such that

$$\mathrm{Op}^w_\hbar(b)\psi(x) = (2\pi)^{-n} \sum_{\kappa \in \mathbb{Z}^n} \int_{\mathbb{T}^n} e^{i\langle x-y,\kappa\rangle} \sigma(\hbar, y, \kappa)\psi(y) dy. \tag{2.25}$$ rep-Op

where in particular $\sigma(\hbar,y,\kappa) = b(y, \frac{\hbar}{2}\kappa) + O(\hbar)$ in $S^m(\mathbb{T}^n \times \mathbb{Z}^n)$. By Theorem 4.3 in [25], it follows the existence of $\widehat{a\sharp b}(\hbar,\cdot) \in S^{\ell+m}(\mathbb{T}^n \times \mathbb{Z}^n)$ such that

$$\mathrm{Op}^w_\hbar(a) \circ \mathrm{Op}^w_\hbar(b)\psi(x) = (2\pi)^{-n} \sum_{\kappa \in \mathbb{Z}^n} \int_{\mathbb{T}^n} e^{i\langle x-y,\kappa\rangle} \widehat{a\sharp b}(\hbar, y, \kappa)\psi(y) dy \tag{2.26}$$

with $\widehat{a\sharp b}(\hbar,y,\kappa) = a \cdot b(y, \frac{\hbar}{2}\kappa) + O(\hbar)$ in $S^{\ell+m}(\mathbb{T}^n \times \mathbb{Z}^n)$. Now apply this operator on $T_x^{-1} \circ T_x\psi$, use again Theorems 8.4 and 4.2 in order to get

$$\mathrm{Op}^w_\hbar(a) \circ \mathrm{Op}^w_\hbar(b) = \mathrm{Op}^w_\hbar(\widetilde{a\sharp b}) \tag{2.27}$$

where $\widetilde{a\sharp b}(\hbar,y,\kappa) = a \cdot b(y,\kappa) + O(\hbar)$ in $S^{\ell+m}(\mathbb{T}^n \times \mathbb{Z}^n)$. By Theorem 5.2 in [25] we get an euclidean simbol $a\sharp b \in S^{\ell+m}(\mathbb{T}^n \times \mathbb{R}^n)$ which is an extention of $\widetilde{a\sharp b}$ modulo $S^{-\infty}(\mathbb{T}^n \times \mathbb{R}^n)$, and thus such that

$$\mathrm{Op}^w_\hbar(a) \circ \mathrm{Op}^w_\hbar(b) = \mathrm{Op}^w_\hbar(a\sharp b) \tag{2.28}$$

where $a\sharp b(\hbar,y,\kappa) = a \cdot b(y,\kappa) + O(\hbar)$ but now in $S^{\ell+m}(\mathbb{T}^n \times \mathbb{R}^n)$. By looking at the second order asymptotics of the simbols, it follows $a\sharp b - b\sharp a = -i\hbar\{a,b\} + O(\hbar^2)$ in $S^{\ell+m-1}(\mathbb{T}^n \times \mathbb{R}^n)$, and this gives (2.22). □

2.1.3 Wigner measures

wig1

To begin, let us recall that in the framework of the usual Weyl quantization on $\mathbb{R}^n$ it can be considered the following space of test functions (see for example [2], [18])

$$\mathcal{A} := \{\varphi \in C_0(\mathbb{R}^n_x \times \mathbb{R}^n_\xi) \mid \|\varphi\|_{\mathcal{A}} := \int_{\mathbb{R}^n} \sup_{x \in \mathbb{R}^n} |\mathcal{F}_\xi \varphi(x,z)|\ dz < +\infty\} \tag{2.29}$$

where $C_0(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ denotes the set of continuous functions tending to zero at infinity, and $\mathcal{F}_\xi$ is the usual Fourier transform in the frequency variables, i.e. $\mathcal{F}_\xi\varphi(x,z) := \int_{\mathbb{R}^n} e^{-i\xi\cdot z}\varphi(x,\xi)d\xi$. In particular, $\mathcal{A}$ is a Banach space and it is a dense subset of $C_0(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$. Hence, its dual space $\mathcal{A}'$ contains $C_0'(\mathbb{R}^n_x \times \mathbb{R}^n_\xi) = \mathcal{M}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ the space of not necessarily nonnegative Radon measures on $\mathbb{R}^{2n}$ of finite mass. As shown in Proposition III.1 of [18], it holds the inequality

$$\Big| \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} W_\hbar\psi_\hbar(x,\xi)\varphi(x,\xi) dx d\xi \Big| \leq (2\pi)^{-n}\|\varphi\|_{\mathcal{A}} \cdot \|\psi_\hbar\|_{L^2}, \tag{2.30}$$ estW18

and hence for any family of wave functions such that $\|\psi_\hbar\|_{L^2(\mathbb{R}^n)} \leq C$ there exists a sequence $\hbar_j \longrightarrow 0^+$ as $j \longrightarrow +\infty$ such that $W_{\hbar_j}\psi_{\hbar_j}$ is converging in $\mathcal{A}'$ to some $W \in \mathcal{A}'$ (thanks Banach–Alaoglu theorem). Moreover, through the use of Husimi transform, it can be proved that in fact any such limit $W \in \mathcal{A}'$ fulfills also $W \in \mathcal{M}^+(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$, i.e. positive Radon measures of finite mass.

We underline that there is an estimate analogous to (2.30) for our toroidal framework which takes the form

$$\Big| \sum_{\eta \in \frac{\hbar}{2}\mathbb{Z}^n} \int_{\mathbb{T}^n} W_\hbar\psi_\hbar(x,\eta) g(x,\eta) dx \Big| \leq (2\pi)^{-n} \sup_{(x,\eta)\in\mathbb{T}^n\times\mathbb{R}^n} |g(x,\eta)| \cdot \|\psi_\hbar\|_{L^2} \tag{2.31}$$ estW76

for all continuous bounded functions $g : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$. Indeed, we observe that for states $\psi_\hbar \in L^2(\mathbb{T}^n)$, by writing the Fourier series $\psi_\hbar(x) = \sum_{\alpha\in\mathbb{Z}^n} \widehat{\psi}_{\hbar,\alpha}\ e^{i\langle x,\alpha\rangle}$ we have

(i) $\displaystyle\sum_{\eta \in \frac{\hbar}{2}\mathbb{Z}^n} W_\hbar\psi_\hbar(x,\eta) = |\psi_\hbar(x)|^2,$

(ii) $\displaystyle(2\pi)^{-n} \int_{\mathbb{T}^n} W_\hbar\psi_\hbar(x,\eta) dx = \begin{cases} |\widehat{\psi}_{\hbar,\alpha}|^2 & \text{when } \eta = \hbar\alpha, \quad \alpha \in \mathbb{Z}^n, \\ 0 & \text{otherwise.} \end{cases}$

Hence, by property (ii) it follows the estimate (2.31).
In view of the above observations, we can now introduce the following

def-wc **Definition 2.5 (Test functions).** *Let $C_0(\mathbb{T}^n_x \times \mathbb{R}^n_\eta)$ be the set of real valued continuous functions on $\mathbb{T}^n_x \times \mathbb{R}^n_\eta$ tending to zero at infinity in η-variables. We consider the subset of those $\phi \in C_0(\mathbb{T}^n_x \times \mathbb{R}^n_\eta)$ that admit the phase space Fourier representation $\phi = F(\widehat{\phi})$ as in (2.16) for some compactly supported $\widehat{\phi} : \mathbb{Z}^n \times \mathbb{R}^n \longrightarrow \mathbb{C}$. We define the set*

$$A := \overline{\Big\{\phi \in C_0(\mathbb{T}^n_x \times \mathbb{R}^n_\eta) \mid \mathrm{supp}(\widehat{\phi}) \text{ is compact}\Big\}}^{L^\infty}. \tag{2.32}$$

Notice that A is a closed linear subset of $L^\infty(\mathbb{T}^n_x \times \mathbb{R}^n_\eta)$ hence it becomes a Banach space when equipped by the L^∞-norm. We also underline that for any fixed $\phi \in C_0(\mathbb{T}^n_x \times \mathbb{R}^n_\eta)$ such that $\mathrm{supp}(\widehat{\phi})$ is compact then ϕ is necessarily a C^∞ function rapidly decreasing in η-variables, and hence we can directly deal with the set of C^∞ functions vanishing at infinity in the η-variables $C^\infty_0(\mathbb{T}^n_x \times \mathbb{R}^n_\eta)$. Thus, we can write

$$A = \overline{\Big\{\phi \in C^\infty_0(\mathbb{T}^n_x \times \mathbb{R}^n_\eta) \mid \mathrm{supp}(\widehat{\phi}) \text{ is compact}\Big\}}^{L^\infty}. \tag{2.33}$$ setA2

Moreover, we easily see that $A \subset C_b(\mathbb{T}^n \times \mathbb{R}^n)$.

We are now in the position to provide the

def-wc1 **Definition 2.6 (Wigner measures).** *Let us fix $\{\psi_\hbar\}_{0<\hbar\leq 1} \in L^2(\mathbb{T}^n)$ with $\|\psi_\hbar\|_{L^2} \leq C$ $\forall 0 < \hbar \leq 1$. We say that $dw \in \mathcal{M}(\mathbb{R}^n_x \times \mathbb{R}^n_\eta)$ is the Wigner measure of the sequence $\{\psi_\hbar\}_{0<\hbar\leq 1}$ if $\forall \phi \in A$*

$$\sum_{\eta \in \frac{\hbar}{2}\mathbb{Z}^n} \int_{\mathbb{T}^n} \phi(x,\eta) W_\hbar\psi_\hbar(x,\eta) dx \longrightarrow \int_{\mathbb{T}^n\times\mathbb{R}^n} \phi(x,\eta) dw(x,\eta) \tag{2.34}$$ med

for some sequence $\hbar = \hbar_j \ \longrightarrow 0^+$ as $j \longrightarrow +\infty$.

Remark 2.7. *The Wigner transform of $\psi_\hbar \in C^\infty(\mathbb{T}^n)$*

$$W_\hbar\psi_\hbar(x,\eta) := (2\pi)^{-n} \int_{\mathbb{T}^n} e^{2\frac{i}{\hbar}\langle z,\eta\rangle} \psi_\hbar(x-z)\psi^\star_\hbar(x+z) dz, \quad \eta \in \frac{\hbar}{2}\mathbb{Z}^n, \tag{2.35}$$ WignerT23

can be rewritten, when acting on test functions ϕ, as

$$\sum_{\eta\in\frac{\hbar}{2}\mathbb{Z}^n}\int_{\mathbb{T}^n}\phi(x,\eta)W_\hbar\psi_\hbar(x,\eta)dx = \sum_{\kappa\in\mathbb{Z}^n}\int_{\mathbb{T}^n}\phi\Big(x,\frac{2}{\hbar}\kappa\Big)W_\hbar\psi_\hbar\Big(x,\frac{2}{\hbar}\kappa\Big)dx, \tag{2.36}$$

$$W_\hbar\psi_\hbar\Big(x,\frac{2}{\hbar}\kappa\Big) = (2\pi)^{-n}\int_{\mathbb{T}^n}e^{i\langle z,\kappa\rangle}\psi_\hbar(x-z)\psi_\hbar^\star(x+z)dz, \quad \kappa\in\mathbb{Z}^n. \tag{2.37}$$

Thus, we notice the $2\pi\mathbb{Z}^n$ - periodicity properties

$$W_\hbar\psi_\hbar\Big(x,\frac{2}{\hbar}(\kappa+2\pi\alpha)\Big) = W_\hbar\psi_\hbar\Big(x,\frac{2}{\hbar}\kappa\Big) \quad \forall\alpha\in\mathbb{Z}^n, \tag{2.38}$$

$$W_\hbar\psi_\hbar\Big(x+2\pi\alpha,\frac{2}{\hbar}\kappa\Big) = W_\hbar\psi_\hbar\Big(x,\frac{2}{\hbar}\kappa\Big) \quad \forall\alpha\in\mathbb{Z}^n. \tag{2.39}$$

From (2.35) we also easily obtain the estimate

$$\sup_{\eta\in\frac{\hbar}{2}\mathbb{Z}^n}\sup_{x\in\mathbb{T}^n}|W_\hbar\psi_\hbar(x,\eta)|\leq(2\pi)^{-n}\|\psi_\hbar\|^2_{L^2}. \tag{2.40}$$ sup-W

Notice that if $\eta\notin\frac{\hbar}{2}\mathbb{Z}^n$ then (2.35) is not defined, since we are computing the integral over the torus and thus we need the $2\pi\mathbb{Z}^n$ periodicity with respect to x-variables of the function within the integral. For this reason, we cannot regard $W_\hbar\psi_\hbar(x,\eta)$ as a wellposed function belonging to $L^\infty(\mathbb{T}^n_x\times\mathbb{R}^n_\eta)$ even if we exhibited the estimate (2.40). This is one of the main differences with the Weyl quantization on $\mathbb{R}^n$ where the Wigner transform $W_\hbar\psi_\hbar(x,\xi)$, when $\psi_\hbar\in L^2(\mathbb{R}^n)$, is a well defined function in $L^\infty(\mathbb{R}^n_x\times\mathbb{R}^n_\xi)$ for any $\hbar>0$.

In the toroidal framework of this paper, under the general assumption $\|\psi_\hbar\|_{L^2}\leq C$ with $C>0$ independent of $\hbar$ we obtain semiclassical limits in A' (see Lemma 2.8) and for suitably defined wave functions (as for example the WKB ones shown in Section 4.2) we can recover semiclassical limits as probability measures on $\mathbb{T}^n\times\mathbb{R}^n$.

t-depW **Lemma 2.8.** *Let $\{\psi_\hbar(t)\}_{0<\hbar\leq1}$ a sequence in $C([-T,T];L^2(\mathbb{T}^n))$ such that $\|\psi_\hbar(t)\|_{L^2}\leq C_T$ for all $t\in[-T,T]$ and $0<\hbar\leq1$. Then, there is a sequence $\hbar_j\longrightarrow0^+$ as $j\longrightarrow+\infty$ such that $W_{\hbar_j}\psi_{\hbar_j}\rightharpoonup W$ in $L^\infty([-T,+T];A')$ with A as in Def 2.5.*

Proof. Since we are assuming $\psi_\hbar\in C([-T,T];L^2(\mathbb{T}^n))$ with $\|\psi_\hbar(t)\|_{L^2}\leq C_T$ then the estimate (2.31) implies that for $0<\hbar\leq1$, the family $W_\hbar\psi_\hbar$ is bounded in $L^\infty([-T,+T];A')$. However, $L^\infty([-T,+T];A')$ is the dual of the separable space $L^1([-T,+T];A)$ and hence the application of the Banach-Alaoglu Theorem provides the existence of a converging subsequence $W_{\hbar_j}\psi_{\hbar_j}\rightharpoonup W$ in $L^\infty([-T,+T];A')$. □

We devote now our attention on the following (locally finite) Borel complex measure on $\mathbb{T}^n\times\mathbb{R}^n$. Let $\mathcal{X}_\Omega$ be the characteristic function of a Borel set $\Omega\subseteq\mathbb{T}^n\times\mathbb{R}^n$, we define

$$\mathbb{P}_\hbar(\Omega):=\sum_{\eta\in\frac{\hbar}{2}\mathbb{Z}^n}\int_{\mathbb{T}^n}\mathcal{X}_\Omega(x,\eta)W_\hbar\psi_\hbar(x,\eta)dx. \tag{2.41}$$ P0

which is a (complex valued) countably additive set function on the Borel sigma algebra of $\mathbb{T}^n\times\mathbb{R}^n$. In particular, we notice that if $\|\psi_\hbar\|_{L^2}=1$ then $|\mathbb{P}_\hbar(\Omega)|\leq1$ for all $\Omega\subseteq\mathbb{T}^n\times\mathbb{R}^n$ and $|\mathbb{P}_\hbar(\mathbb{T}^n\times\mathbb{R}^n)|=1$. As usual, we say that $\mathbb{P}_\hbar$ is weak (i.e. narrow) convergent to a Borel complex measure $\mathbb{P}$ if $\forall f\in C_b(\mathbb{T}^n\times\mathbb{R}^n)$ it holds

$$\int_{\mathbb{T}^n\times\mathbb{R}^n}f(x,\eta)d\mathbb{P}_\hbar(x,\eta)\longrightarrow\int_{\mathbb{T}^n\times\mathbb{R}^n}f(x,\eta)d\mathbb{P}(x,\eta) \tag{2.42}$$ NC

as $\hbar\longrightarrow0^+$. In fact, since $f\in C_b(\mathbb{T}^n\times\mathbb{R}^n)$, it holds

$$\int_{\mathbb{T}^n\times\mathbb{R}^n}f(x,\eta)d\mathbb{P}_\hbar(x,\eta)=\sum_{\eta\in\frac{\hbar}{2}\mathbb{Z}^n}\int_{\mathbb{T}^n}f(x,\eta)W_\hbar\psi_\hbar(x,\eta)dx. \tag{2.43}$$ NC2

Definition 2.9. *The family of (complex Borel) measures* $\{\mathbb{P}_\hbar\}_{0<\hbar\leq 1}$ *on the probability space* $\mathbb{T}^n \times \mathbb{R}^n$ *(equipped with the Borel sigma algebra) is called tight if*

$$\lim_{R\to+\infty} \sup_{0<\hbar\leq 1} \int_{\mathbb{T}^n\times\{\mathbb{R}^n\setminus B_R\}} d\mathbb{P}_\hbar(x,\eta) = 0. \tag{2.44}$$ tight-C

Thanks to a well-known Prokhorov's Theorem, the set of measures $\{\mathbb{P}_\hbar\}_{0<\hbar\leq 1}$ *is relatively compact with respect to the weak topology if and only if is tight. Notice that the condition (2.44) reads equivalently as* $\lim_{R\to+\infty}\sup_{0<\hbar\leq 1}\mathbb{P}_\hbar(\mathbb{T}^n\times\{\mathbb{R}^n\setminus B_R\}) = 0$.

Remark 2.10. *When* $\mathbb{P}_\hbar = \mathbb{P}_\hbar^\pm$ *is associated to the class of WKB wave functions* $\varphi_\hbar^\pm$ *described in Section 4.2, we will directly prove the weak convergence (with test functions in A) to some meaningful probability measures of monokinetic type (see Theorem 4.8). On the other hand, within Lemma 4.5 we will also prove that such measures* $\mathbb{P}_\hbar^\pm$ *fulfill the tightness condition (2.44), and in this way we can apply the next result on time propagation of tightness. This ensures the existence of the Wigner probability measure associated to the solution of the Schrödinger equation, and its coincidence with the solution of the underlying classical continuity equation, see Theorem 5.1 and Proposition 5.3.*

Prop-T **Proposition 2.11 (Propagation of tightness).** *Let* $H = \frac{1}{2}|\eta|^2 + V(x)$ *with* $V \in C^\infty(\mathbb{T}^n)$, $\psi_\hbar \in L^2(\mathbb{T}^n)$ *be such that* $\|\psi_\hbar\|_{L^2} \leq C$ *for all* $0 < \hbar \leq 1$. *Assume that* $\mathbb{P}_\hbar$ *as in (2.41) is tight. Define* $\psi_\hbar(t) := e^{-\frac{i}{\hbar}\mathrm{Op}_\hbar(H)t}\psi_\hbar$. *Then,* $\mathbb{P}_\hbar(t)$ *is tight for any* $t \in \mathbb{R}$.

Proof. Let $Y \in C^\infty(\mathbb{R}^n_\eta;[0,1])$ be such that $Y(\eta) = 1$ on $|\eta| > 1$ and $Y(\eta) = 0$ on $|\eta| < 1/2$; for $R > 0$ define $Y_R(\eta) := Y(\eta/R)$. Then, $|\nabla_\eta Y| \leq C/R$ and $|\nabla^2_\eta Y| \leq C/R^2$ for some $C > 0$. In fact, we can regard $Y \in C_b^\infty(\mathbb{T}^n_x \times \mathbb{R}^n_\eta;[0,1])$.

$$\frac{d}{ds}\langle\psi_\hbar(s),\mathrm{Op}_\hbar(Y_R)\psi_\hbar(s)\rangle_{L^2} = \frac{i}{\hbar}\langle\psi_\hbar(s),[\mathrm{Op}_\hbar(Y_R),\mathrm{Op}_\hbar(H)]\psi_\hbar(s)\rangle_{L^2}. \tag{2.45}$$

Recalling Theorem 2.4, the commutator reads $[\mathrm{Op}_\hbar(Y_R),\mathrm{Op}_\hbar(H)] = \mathrm{Op}_\hbar(\{Y_R,H\}_M)$ where the Moyal bracket has the asymptotics $\{Y_R,H\}_M = -i\hbar\{Y_R,H\} + D_\hbar$ in $S^2(\mathbb{T}^n\times\mathbb{R}^n)$ where the remainder $D_\hbar \simeq O(\hbar^2)$ and involves the second order derivatives of Y_R and H. But $|\partial_x^\alpha\partial_\eta^\beta H(z)| \leq c_1$ and $|\partial_x^\alpha\partial_\eta^\beta Y_R(z)| \leq c_2/R^2$ for $|\alpha+\beta| = 2$; hence $|D_\hbar| \simeq R^{-2}$ as $R \longrightarrow +\infty$ (uniformly on $\hbar$). Moreover $\{Y_R,H\}(z) = \partial_x Y_R\partial_\eta H - \partial_\eta Y_R\partial_x H = -\partial_\eta Y_R\partial_x H$ hence $|\{Y_R,H\}(z)| \leq c_3/R$. By recalling the L^2 - boundedness of the Weyl operators with simbols in $S^0_{0,0}(\mathbb{T}^n\times\mathbb{R}^n)$ as shown in Theorem 2.3 and using the assumption $\|\psi_\hbar\|_{L^2} \leq C$, we deduce that

$$\Big|\frac{d}{ds}\langle\psi_\hbar(s),\mathrm{Op}_\hbar(Y_R)\psi_\hbar(s)\rangle_{L^2}\Big| \leq K\cdot R^{-1} \tag{2.46}$$

for some $K > 0$ independent on $\hbar$ and t. Thus

$$\langle\psi_\hbar(t),\mathrm{Op}_\hbar(Y_R)\psi_\hbar(t)\rangle_{L^2} = \langle\psi_\hbar(0),\mathrm{Op}_\hbar(Y_R)\psi_\hbar(0)\rangle_{L^2} + \int_0^t \frac{d}{ds}\langle\psi_\hbar(s),\mathrm{Op}_\hbar(Y_R)\psi_\hbar(s)\rangle_{L^2}ds \tag{2.47}$$

and

$$\begin{aligned} |\langle\psi_\hbar(t),\mathrm{Op}_\hbar(Y_R)\psi_\hbar(t)\rangle_{L^2}| &\leq |\langle\psi_\hbar(0),\mathrm{Op}_\hbar(Y_R)\psi_\hbar(0)\rangle_{L^2}| + \Big|\int_0^t \frac{d}{ds}\langle\psi_\hbar(s),\mathrm{Op}_\hbar(Y_R)\psi_\hbar(s)\rangle_{L^2}ds\Big| \\ &\leq |\langle\psi_\hbar(0),\mathrm{Op}_\hbar(Y_R)\psi_\hbar(0)\rangle_{L^2}| + t\,K\cdot R^{-1} \qquad (2.48) \end{aligned}$$

Notice that, from the property (ii) of $W_\hbar\psi_\hbar$, it follows

$$\begin{aligned} \langle\psi_\hbar,\mathrm{Op}_\hbar(Y_R)\psi_\hbar\rangle_{L^2} &= \sum_{\eta\in\frac{\hbar}{2}\mathbb{Z}^n}\int_{\mathbb{T}^n} W_\hbar\psi_\hbar(x,\eta)Y_R(\eta)dx \qquad (2.49) \\ &= \sum_{\eta\in\frac{\hbar}{2}\mathbb{Z}^n} Y_R(\eta)\int_{\mathbb{T}^n} W_\hbar\psi_\hbar(x,\eta)dx = \sum_{\alpha\in\mathbb{Z}^n} Y_R(\hbar\alpha)|\widehat{\psi}_{\hbar,\alpha}|^2, \qquad (2.50) \end{aligned}$$

thus any term of the series is non negative. The same holds true for

$$\mathbb{P}_\hbar(\mathbb{T}^n \times U) = \sum_{\eta \in \frac{\hbar}{2}\mathbb{Z}^n} \int_{\mathbb{T}^n} W_\hbar \psi_\hbar(x,\eta) \mathcal{X}_U(\eta) dx \tag{2.51}$$

$$= \sum_{\eta \in \frac{\hbar}{2}\mathbb{Z}^n} \mathcal{X}_U(\eta) \int_{\mathbb{T}^n} W_\hbar \psi_\hbar(x,\eta) dx = \sum_{\alpha \in \mathbb{Z}^n} \mathcal{X}_U(\hbar\alpha) |\widehat{\psi}_{\hbar,\alpha}|^2, \tag{2.52}$$

where U is any Borel set in $\mathbb{R}^n$.
By defining $M_R := \mathbb{T}^n \times \{\mathbb{R}^n \backslash B_R\}$, and recalling that $Y_R(\eta) = 0$ for $|\eta| < R/2$ whereas $Y_R(\eta) = 1$ for $|\eta| > R$, we can write

$$\mathbb{P}_\hbar(t)(M_R) \leq \langle \psi_\hbar(t), \mathrm{Op}_\hbar(Y_R)\psi_\hbar(t)\rangle_{L^2} \tag{2.53}$$

$$\leq \langle \psi_\hbar(0), \mathrm{Op}_\hbar(Y_R)\psi_\hbar(0)\rangle_{L^2} + t\ K \cdot R^{-1} \tag{2.54}$$

$$\leq \mathbb{P}_\hbar(M_{R/2}) + t\ K \cdot R^{-1} \tag{2.55}$$

and hence (recalling the tightness assumption on $\mathbb{P}_\hbar$)

$$\lim_{R \to +\infty} \sup_{0<\hbar\leq 1} \mathbb{P}_\hbar(t)(M_R) = 0. \tag{2.56}$$

□

2.2 A quick review of weak KAM theory and Aubry-Mather theory

kam1

2.2.1 Weak solutions of Hamilton-Jacobi equation

sub-wHJ

The *weak KAM theory* deals with the existence of Lipschitz continuous solutions of the stationary Hamilton-Jacobi equation

$$H(x, P + \nabla_x v(P,x)) = \bar{H}(P), \quad P \in \mathbb{R}^n, \tag{2.57}$$

def-eff0

for Tonelli Hamiltonians $H \in C^\infty(\mathbb{T}^n \times \mathbb{R}^n; \mathbb{R})$, that is to say, for functions H such that $\eta \mapsto H(x,\eta)$ is strictly convex and uniformly superlinear in the fibers of the canonical projection $\pi : \mathbb{T}^n \times \mathbb{R}^n \longrightarrow \mathbb{T}^n$. The function $\bar{H}(P)$ is called the *effective Hamiltonian* and, as showed in [6] (see also [12]), it can be expressed by the inf-sup formula

$$\bar{H}(P) = \inf_{v \in C^\infty(\mathbb{T}^n;\mathbb{R})} \sup_{x \in \mathbb{T}^n} H(x, P + \nabla_x v(x)) \tag{2.58}$$

def-eff

which is a convex function of $P \in \mathbb{R}^n$ (hence continuous). The Lax-Oleinik semigroup of negative and positive type is defined as

$$T_t^{\mp} u(x) := \inf_\gamma \left\{ u(\gamma(0)) \pm \int_0^t L(\gamma(s), \dot{\gamma}(s)) - P \cdot \dot{\gamma}(s)\ ds \right\},$$

where the infimum is taken over all absolutely continuous curves $\gamma : [0,t] \longrightarrow \mathbb{T}^n$ such that $\gamma(t) = x$. A function $v_- \in C^{0,1}(\mathbb{T}^n; \mathbb{R})$ is said to be a *weak KAM solution of negative type* for (2.57) if $\forall\ t \geq 0$

$$T_t^- v_- = v_- - t\, \bar{H}(P), \tag{2.59}$$

back-

whereas it is said to be a *weak KAM solution of positve type* if $\forall\ t \geq 0$

$$T_t^+ v_+ = v_+ + t\, \bar{H}(P). \tag{2.60}$$

back+

As a consequence, for any weak KAM solution it holds

$$\overline{\text{Graph}(P+\nabla_x v_\pm(P,\cdot))} \subset \{(x,\eta)\in\mathbb{T}^n\times\mathbb{R}^n \mid H(x,\eta)=\bar{H}(P)\} \tag{2.61}$$ inc-G

Geometrically, equations (2.59) and (2.60) imply also that we are looking at functions for which the graphs are invariant under the backward (resp. forward) Euler-Lagrange flow, namely

$$\varphi_H^t\Big(\text{Graph}(P+\nabla_x v_-(P,\cdot))\Big) \subseteq \text{Graph}(P+\nabla_x v_-(P,\cdot)) \quad \forall t\leq 0 \tag{2.62}$$

and

$$\varphi_H^t\Big(\text{Graph}(P+\nabla_x v_+(P,\cdot))\Big) \subseteq \text{Graph}(P+\nabla_x v_+(P,\cdot)) \quad \forall t\geq 0 \tag{2.63}$$

see Theorems 4.13.2 and 4.13.3 in [13]. Moreover, it is proved that the maps $x\longmapsto(x,P+\nabla_x v_\pm(P,x))$ are continuous on $\text{dom}(\nabla_x v_\pm)$. As showed within Th. 7.6.2 of [13], all the Lipschitz continuous weak KAM solutions of negative type coincide with the so-called *viscosity solutions* in the sense of Crandall-Lions [7].

2.2.2 Mather measures

sec-M

The Aubry-Mather theory proves the existence of invariant and Action-minimizing measures as well as invariant and Action-minimizing sets in the phase space. Here we recall only those results which we are going to use in what follows, and for an exahustive treatment we address the reader to [19], [22], [26].

Recall that a compactly supported Borel probability measure $d\mu$ on the tangent bundle $T\mathbb{T}^n$ is called *invariant* with respect to the Lagrangian flow $\phi^t:\mathbb{T}^n\times\mathbb{R}^n\longrightarrow\mathbb{T}^n\times\mathbb{R}^n$ related to a Lagrangian function $L(x,\xi)$, which we suppose to be Legendre-related to a Tonelli Hamiltonian $H(x,p)$, if

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} f(\phi^t(x,\xi))d\mu(x,\xi)=\int_{\mathbb{T}^n\times\mathbb{R}^n} f(x,\xi)d\mu(x,\xi),$$

for all $t\in\mathbb{R}$ and all $f\in C_0^\infty(\mathbb{T}^n\times\mathbb{R}^n;\mathbb{R}$. Recall also that a Borel probability measure $d\mu$ is said to be *closed* if for every $g\in C^\infty(\mathbb{T}^n;\mathbb{R})$ one has

$$\int_{\mathbb{T}^n\times\mathbb{R}^n}\nabla_x g(x)\cdot\xi\ d\mu(x,\xi)=0.$$

One says that an invariant compactly supported Borel probability measure $d\mu_P$ is a *Mather measure* if it satisfies the Mather *P-minimal problem* for all $P\in\mathbb{R}^n$, that is,

$$\int_{\mathbb{T}^n\times\mathbb{R}^n}\big(\,L(x,\xi)-P\cdot\xi\,\big)\ d\mu_P(x,\xi)=\inf_{d\mu}\int_{\mathbb{T}^n\times\mathbb{R}^n}\big(\,L(x,\xi)-P\cdot\xi\,\big)\ d\mu(x,\xi),$$

where the infimum is taken over all invariant compactly supported Borel probability measures $d\mu$. Moreover, the miminizing value of the Action is related to the effective Hamiltonian as

$$-\bar{H}(P)=\int_{\mathbb{T}^n\times\mathbb{R}^n}\big(\,L(x,\xi)-P\cdot\xi\,\big)\ d\mu_P(x,\xi).$$

It has been also proved that the Mather measures of a Tonelli-Lagrangian are those which minimize the action in the class of all (compactly supported) closed measures (see for example [5]). This fact will be useful in the proof of Theorem 1.2. As for the Mather set, it involves the supports of all Mather's measures, and is defined to be

$$\widetilde{\mathcal{M}}_P:=\overline{\bigcup_{d\mu_P}\text{supp}\ d\mu_P}. \tag{2.64}$$ def-M

We recall that Mather proved in [22] that the set $\widetilde{\mathcal{M}}_P$ is not empty, compact and Lipschitz graphs above $\mathbb{T}^n$, namely the restriction of $\pi : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{T}^n$ to $\widetilde{\mathcal{M}}_P$ is an injective map and $\pi^{-1} : \pi(\widetilde{\mathcal{M}}_P) \to \widetilde{\mathcal{M}}_P$ is Lipschitz. The projected Mather set $\pi(\widetilde{\mathcal{M}}_P)$ is denoted by $\mathcal{M}_P$.
By following the Remark 4.11 in [26], one can take a countably dense set of Mather measures $\{d\mu_{j,P}\}_{j\in\mathbb{N}}$ such that

$$d\bar{\mu}_P := \sum_{j\in\mathbb{N}} d\mu_{j,P} \tag{2.65}$$

is a Mather measure with full support on the Mather set $\widetilde{\mathcal{M}}_P$. For any fixed Mather measure $d\mu_P$, we denote by

$$dw_P := \mathcal{L}_\star(d\mu_P), \qquad d\sigma_P := \pi_\star(dw_P) = \pi_\star(d\mu_P), \tag{2.66}$$

def-dwP

the push forward by the Legendre transform $\mathcal{L}(x,\xi) = (x, \nabla_\xi L(x,\xi))$ and by the canonical projection $\pi(x,\eta) = x$.

sec-Au

2.2.3 Aubry sets

As for the definition of the Aubry sets $\widetilde{\mathcal{A}}_P$ (in the tangent bundle of a manifold) involving regular P-minimizers we refer to [13]; we recall here that its Legendre transform can be given by

$$\mathcal{A}^*_P = \bigcap_{v\in S^{\mp}_P} \Big\{ (x, P + \nabla_x v(P,x)) \mid x \in \mathbb{T}^n \text{ s.t. } \exists\ \nabla_x v(P,x) \Big\} \tag{2.67}$$

def-aubry

where the intersection is taken over all Lipschitz continuous weak KAM solutions $S^{\mp}_P$ of negative (resp. positive) type of the Hamilton-Jacobi equation (2.57). This set is invariant under the Hamiltonian dynamics and one has the meaningful inclusion

$$\mathcal{M}^\star_P := \mathcal{L}(\widetilde{\mathcal{M}}_P) \subseteq \mathcal{A}^*_P. \tag{2.68}$$

inc-MA

The $\mathcal{A}^\star_P$ is compact, the restriction of $\pi : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{T}^n$ to $\mathcal{A}^\star_P$ is an injective map and $\pi^{-1} : \pi(\mathcal{A}^\star_P) \to \mathcal{A}^\star_P$ is Lipschitz (see [13], [26]).

3 The dynamics of the Wigner transform on the torus

wtd1

3.1 The Schrödinger equation on the torus

Let us consider the classical Hamiltonian $H = \frac{1}{2}|\eta|^2 + V(x)$, with $V \in C^\infty(\mathbb{T}^n;\mathbb{R})$. Thus we have $H \in S^2(\mathbb{T}^n \times \mathbb{R}^n)$, namely the simbol class described in (2.8) with $m = 2$. We now consider the Schrödinger equation:

$$\begin{array}{rcl} i\hbar\partial_t\psi_\hbar(t,x) &=& \mathrm{Op}^w_\hbar(H)\psi_\hbar(t,x) \\ \psi_\hbar(0,x) &=& \varphi_\hbar(x) \end{array} \tag{3.69}$$

where $\mathrm{Op}^w_\hbar(H)$ is the Weyl quantization of H as in (2.10). As for the initial datum, we can require $\varphi_\hbar \in W^{2,2}(\mathbb{T}^n;\mathbb{C})$ and $\|\varphi_\hbar\|_{L^2} \leq C\ \forall\ 0 < \hbar \leq 1$. The one parameter group of unitary operators $e^{-\frac{i}{\hbar}\mathrm{Op}^w_\hbar(H)t}$ can be defined on the whole $L^2(\mathbb{T}^n;\mathbb{C})$. In fact, this is because the Schrödinger operator $\hat{H}_\hbar := -\frac{1}{2}\hbar^2\Delta_x + V(x)$ is coinciding with $\mathrm{Op}^w_\hbar(H)$. This is the content of the Lemma 6.1 shown in the Appendix.

3.2 The equation for the Wigner transform

In this section we provide a result on the equation for the Wigner transform of the solution of the Schrödinger equation written on the torus. The well known arguments within the framework of the

Weyl quantization on $\mathbb{R}^n$ (see [2], [3], [18]) must be adapted for the Weyl quantization on $\mathbb{T}^n$.

The first result reads as follows

TH21 **Proposition 3.1.** *Let $\psi_\hbar$ be the solution of (3.69), and $f \in C^\infty([0,t]\times\mathbb{T}^n\times\mathbb{R}^n;\mathbb{R})$ such that $\forall s \in [0,t]$ it holds $f(s,\cdot) \in A$ as in Def 2.5. Then,*

$$\int_0^t \sum_{\eta\in\frac{\hbar}{2}\mathbb{Z}^n} \int_{\mathbb{T}^n} \Big[\Big(\partial_s f + \eta\cdot\nabla_x f\Big)(s,x,\eta) W_\hbar\psi_\hbar(s,x,\eta) + f(s,x,\eta)\mathcal{E}_\hbar\psi_\hbar(s,x,\eta)\Big] dx ds = 0 \tag{3.70}$$ w-trans

where

$$\mathcal{E}_\hbar\psi_\hbar(s,x,\eta) := \frac{i}{(2\pi)^n\hbar}\int_{\mathbb{T}^n} e^{2\frac{i}{\hbar}\langle z,\eta\rangle}\{V(x+z)-V(x-z)\}\psi_\hbar(s,x-z)\overline{\psi}_\hbar(s,x+z)dz. \tag{3.71}$$

Proof. We interpret all the subsequent partial derivatives in the distributional sense of A'. To begin,

$$\begin{aligned}\partial_t W_\hbar\psi &= (2\pi)^{-n}\int_{\mathbb{T}^n} e^{2\frac{i}{\hbar}\langle z,\eta\rangle}\partial_t\psi_\hbar(t,x-z)\overline{\psi}_\hbar(t,x+z)dz \\ &+ (2\pi)^{-n}\int_{\mathbb{T}^n} e^{2\frac{i}{\hbar}\langle z,\eta\rangle}\psi_\hbar(t,x-z)\partial_t\overline{\psi}_\hbar(t,x+z)dz. \end{aligned} \tag{3.72}$$

Since $\psi_\hbar$ solves the Schrödinger equation, it follows

$$\begin{aligned} &\partial_t\psi_\hbar(t,x-z)\overline{\psi}(t,x+z) + \psi_\hbar(t,x-z)\partial_t\overline{\psi}_\hbar(t,x+z) && (3.73)\\ =\ &\frac{i\hbar}{2}[(\Delta_x\psi_\hbar(t,x-z))\overline{\psi}_\hbar(t,x+z) - \psi_\hbar(t,x-z)\Delta_x\overline{\psi}_\hbar(t,x+z)] && (3.74)\\ +\ &i\hbar^{-1}[V(x+z)-V(x-z)]\psi_\hbar(t,x-z)\overline{\psi}_\hbar(t,x+z). \end{aligned}$$

Now recall the simple equality $(\Delta_x f)g - f\Delta_x g = \mathrm{div}_x[(\nabla_x f)g - f\nabla_x g]$, so that

$$\begin{aligned} &(\Delta_x\psi_\hbar(t,x-z))\overline{\psi}_\hbar(t,x+z) - \psi_\hbar(t,x-z)\Delta_x\overline{\psi}_\hbar(t,x+z)\\ =\ &2\,\mathrm{div}_x\nabla_z[\psi_\hbar(t,x-z)\overline{\psi}_\hbar(t,x+z)]. \end{aligned} \tag{3.75}$$

Then, insert (3.75) in (3.74), so that

$$\begin{aligned} &\partial_t\psi_\hbar(t,x-z)\overline{\psi}(t,x+z) + \psi_\hbar(t,x-z)\partial_t\overline{\psi}_\hbar(t,x+z) && (3.76)\\ =\ &i\hbar\,\mathrm{div}_x\nabla_z[\psi_\hbar(t,x-z)\overline{\psi}_\hbar(t,x+z)] + \frac{i}{\hbar}\,[V(x+z)-V(x-z)]\psi_\hbar(t,x-z)\overline{\psi}_\hbar(t,x+z) && (3.77) \end{aligned}$$

Moreover, an easy computation involving integration by parts shows

$$\eta\cdot\nabla_x W_\hbar\psi_\hbar = -i\hbar\,(2\pi)^{-n}\int_{\mathbb{T}^n} e^{2\frac{i}{\hbar}\langle z,\eta\rangle}\mathrm{div}_z\nabla_x[\psi_\hbar(t,x-z)\overline{\psi}(t,x+z)]dz. \tag{3.78}$$ W-eta

Hence, by (3.77) and (3.78) we directly get the statement. □

reg-delta **Lemma 3.2.** *Let $\epsilon > 0$ and $g(\epsilon,\cdot) : \mathbb{T}^n \longrightarrow \mathbb{R}^+$ defined as*

$$g(\epsilon,y) := \frac{1}{(2\pi)^n}\sum_{\kappa_0\in\mathbb{Z}^n} e^{-\epsilon|\kappa_0|^2}e^{-i\langle y,\kappa_0\rangle} = \frac{1}{(2\pi)^n}\sum_{\xi\in\mathbb{Z}^n}\Big(\frac{\pi}{\epsilon}\Big)^{\frac{n}{2}} e^{-|\xi-y|^2(4\epsilon)^{-1}}. \tag{3.79}$$

Then, $\forall\psi \in C^\infty(\mathbb{T}^n;\mathbb{C})$

$$\lim_{\epsilon\to0^+}\int_{\mathbb{T}^n} g(\epsilon,y-y_0)\psi(y_0)dy_0 = \psi(y). \tag{3.80}$$

Proof. Let $G(\kappa_0, \epsilon, y) := e^{-\epsilon|\kappa_0|^2} e^{-i\langle y,\kappa_0\rangle}$, then $\widehat{G}(\xi, \epsilon, y) := \int_{\mathbb{R}^n} e^{-i\langle \xi,\kappa_0\rangle} G(\kappa_0, \epsilon, y) d\kappa_0$ reads

$$\widehat{G}(\xi, \epsilon, y) = \Big(\frac{\pi}{\epsilon}\Big)^{\frac{n}{2}} e^{-|\xi-y|^2(4\epsilon)^{-1}}$$

By applying the Poisson's summation formula (see for example D-K [8]),

$$g(\epsilon, y) = \frac{1}{(2\pi)^n} \sum_{\xi\in\mathbb{Z}^n} \Big(\frac{\pi}{\epsilon}\Big)^{\frac{n}{2}} e^{-|\xi-y|^2(4\epsilon)^{-1}} = \frac{1}{(2\pi)^n} \sum_{\xi\in\mathbb{Z}^n} \Big(\frac{\pi}{\epsilon}\Big)^{\frac{n}{2}} e^{-|2\pi\xi-2\pi y|^2(16\pi^2\epsilon)^{-1}}. \tag{3.81}$$ ge2

Now recall the identification $\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n$, fix the periodicity domain $y_0 \in Q_n := [0, 2\pi]^n$, so that

$$\lim_{\epsilon\to 0^+} \int_{Q_n} g(\epsilon, y - y_0)\psi(y_0)dy_0 \tag{3.82}$$

$$= \lim_{\epsilon\to 0^+} \Big(\frac{1}{4\pi\epsilon}\Big)^{\frac{n}{2}} \sum_{\xi\in\mathbb{Z}^n} \int_{\mathbb{R}^n} e^{-|2\pi\xi-2\pi(y-y_0)|^2(16\pi^2\epsilon)^{-1}} \psi(y_0)\mathcal{X}_{Q_n}(y_0)dy_0 \tag{3.83}$$

$$= \lim_{\epsilon\to 0^+} \Big(\frac{1}{4\pi\epsilon}\Big)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-|y-y_0|^2(4\epsilon)^{-1}} \psi(y_0)\mathcal{X}_{Q_n}(y_0)dy_0 = \psi(y). \tag{3.84}$$

□

In the following, we provide the evolution equation for the Wigner transform $W_\hbar\psi_\hbar$ of the solution of the Schrödinger's equation on the torus,

$$\partial_t W_\hbar\psi_\hbar + \eta\cdot\nabla_x W_\hbar\psi_\hbar + \mathcal{K}_\hbar \star_\eta W_\hbar\psi_\hbar = 0 \tag{3.85}$$

written in the distributional sense. More precisely, $\forall f \in C^\infty([0, t] \times \mathbb{T}^n \times \mathbb{R}^n; \mathbb{R})$ such that $f(s, \cdot) \in A$ $\forall s \in [0, t]$ as in Def 2.5 it holds

$$\int_0^t \sum_{\eta\in\frac{\hbar}{2}\mathbb{Z}^n} \int_{\mathbb{T}^n} \Big[\Big(\partial_s f + \eta\cdot\nabla_x f\Big)(s, x, \eta) W_\hbar\psi_\hbar(s, x, \eta) + f(s, x, \eta)\ \mathcal{K}_\hbar \star_\eta W_\hbar\psi_\hbar(s, x, \eta)\Big] dxds = 0 \tag{3.86}$$ w-trans01

where for $\eta \in \frac{\hbar}{2}\mathbb{Z}^n$

$$\mathcal{K}_\hbar(s, x, \eta) := \frac{i}{(2\pi)^n\hbar} \int_{\mathbb{T}^n} e^{2\frac{i}{\hbar}\langle z,\eta\rangle} \{V(x+z) - V(x-z)\} dz, \tag{3.87}$$

$$\mathcal{K}_\hbar \star_\eta W_\hbar\psi_\hbar(s, x, \eta) := \sum_{\kappa_0\in\mathbb{Z}^n} \mathcal{K}_\hbar\Big(s, x, \eta - \frac{\hbar}{2}\kappa_0\Big) W_\hbar\psi_\hbar\Big(s, x, \frac{\hbar}{2}\kappa_0\Big). \tag{3.88}$$

TH22 **Theorem 3.3.** *Let $\psi_\hbar$ be the solution of (3.69). Then, it holds*

$$\partial_t W_\hbar\psi_\hbar + \eta\cdot\nabla_x W_\hbar\psi_\hbar + \mathcal{K}_\hbar \star_\eta W_\hbar\psi_\hbar = 0 \tag{3.89}$$ w-trans

in the distributional sense as in (3.86).

Proof. We exhibit a short proof based on the previous result, namely we simply show that convolution (3.88) is well defined and coincides with the remainder term (3.71). Since $V \in C^\infty(\mathbb{T}^n; \mathbb{R})$, the related Fourier components $V_\omega := (2\pi)^{-n} \int_{\mathbb{T}^n} e^{i\omega z} V(z)dz$, $\omega \in \mathbb{Z}^n$, fulfill $|V_\omega| \le c_j\langle\omega\rangle^j$ $\forall j \in \mathbb{N}$ and some $c_j > 0$. An easy computation shows that

$$\mathcal{K}_\hbar\Big(s, x, \frac{\hbar}{2}\kappa\Big) = \frac{i}{(2\pi)^n\hbar}(e^{-i\kappa\cdot x}V_\kappa - e^{+i\kappa\cdot x}V^\star_\kappa), \quad \kappa \in \mathbb{Z}^n. \tag{3.90}$$

Moreover, $\|W_\hbar\psi_\hbar(s,\cdot)\|_\infty \leq (2\pi)^{-n}C^2\ \forall s \in \mathbb{R}$. Thus, the series in (3.88) is absolutely convergent, and we can write down the regularization (useful in the subsequent computations):

$$\mathcal{K}_\hbar \star_\eta W_\hbar\psi_\hbar \quad = \quad \lim_{\epsilon\to 0^+} \sum_{\kappa_0\in\mathbb{Z}^n} e^{-\epsilon|\kappa_0|^2}\mathcal{K}_\hbar\Big(s,x,\eta-\frac{\hbar}{2}\kappa_0\Big)W_\hbar\psi_\hbar\Big(s,x,\frac{\hbar}{2}\kappa_0\Big). \tag{3.91}$$

We look at the regularization:

$$\sum_{\kappa_0\in\mathbb{Z}^n} e^{-\epsilon|\kappa_0|^2}\mathcal{K}_\hbar\Big(s,x,\eta-\frac{\hbar}{2}\kappa_0\Big)W_\hbar\psi_\hbar\Big(s,x,\frac{\hbar}{2}\kappa_0\Big) \tag{3.92}$$

$$= \sum_{\kappa_0\in\mathbb{Z}^n} e^{-\epsilon|\kappa_0|^2}\frac{i}{(2\pi)^n\hbar}\int_{\mathbb{T}^n} e^{2\frac{i}{\hbar}\langle z,\eta-\frac{\hbar}{2}\kappa_0\rangle}\{V(x+z)-V(x-z)\}dz \tag{3.93}$$

$$\times \quad \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} e^{2\frac{i}{\hbar}\langle\tilde{z},\frac{\hbar}{2}\kappa_0\rangle}\psi_\hbar(s,x-\tilde{z})\psi_\hbar^\star(s,x+\tilde{z})d\tilde{z}$$

$$= \frac{i}{(2\pi)^n\hbar}\int_{\mathbb{T}^n}\int_{\mathbb{T}^n} e^{2\frac{i}{\hbar}\langle z,\eta\rangle}\Big[\frac{1}{(2\pi)^n}\sum_{\kappa_0\in\mathbb{Z}^n} e^{-\epsilon|\kappa_0|^2}e^{-i\langle z-\tilde{z},\kappa_0\rangle}\Big] \tag{3.94}$$

$$\times \quad \{V(x+z)-V(x-z)\}\psi_\hbar(s,x-\tilde{z})\psi_\hbar^\star(s,x+\tilde{z})dzd\tilde{z}$$

However, for any fixed $\epsilon > 0$, the function

$$g(\epsilon, z-\tilde{z}) := \frac{1}{(2\pi)^n}\sum_{\kappa_0\in\mathbb{Z}^n} e^{-\epsilon|\kappa_0|^2}e^{-i\langle z-\tilde{z},\kappa_0\rangle} \tag{3.95}$$

defines a tempered distribution on $C^\infty(\mathbb{T}^n;\mathbb{C})$ converging to $\delta(z-\tilde{z})$ as $\epsilon\to 0^+$ (see Lemma 3.2). To conclude,

$$\mathcal{K}_\hbar \star_\eta W_\hbar\psi_\hbar$$

$$= \lim_{\epsilon\to 0^+}\frac{i}{(2\pi)^n\hbar}\int_{\mathbb{T}^n}\int_{\mathbb{T}^n} e^{2\frac{i}{\hbar}\langle z,\eta\rangle}g(\epsilon,z-\tilde{z})\{V(x+z)-V(x-z)\}\psi_\hbar(s,x-\tilde{z})\psi_\hbar^\star(s,x+\tilde{z})dzd\tilde{z}$$

$$= \frac{i}{(2\pi)^n\hbar}\int_{\mathbb{T}^n}\lim_{\epsilon\to 0^+}\int_{\mathbb{T}^n} e^{2\frac{i}{\hbar}\langle z,\eta\rangle}g(\epsilon,z-\tilde{z})\{V(x+z)-V(x-z)\}\psi_\hbar(s,x-\tilde{z})\psi_\hbar^\star(s,x+\tilde{z})dzd\tilde{z}$$

$$= \frac{i}{(2\pi)^n\hbar}\int_{\mathbb{T}^n} e^{2\frac{i}{\hbar}\langle\tilde{z},\eta\rangle}\{V(x+\tilde{z})-V(x-\tilde{z})\}\psi_\hbar(s,x-\tilde{z})\psi_\hbar^\star(s,x+\tilde{z})d\tilde{z} =: \mathcal{E}_\hbar\psi_\hbar.$$

□

4 Semiclassical limits of Wigner transforms on the torus

scwt

4.1 The Liouville equation

This section is devoted to the Liouville equation written in the measure sense on $\mathbb{T}^n \times \mathbb{R}^n$ solved by the semiclassical asymptotics of the toroidal Wigner transform.

TH41 **Theorem 4.1.** *Let $\psi_\hbar(t) := e^{-\frac{i}{\hbar}\mathrm{Op}^w_\hbar(H)t}\varphi_\hbar$ where $\varphi_\hbar \in L^2(\mathbb{T}^n;\mathbb{C})$ and $\|\varphi_\hbar\|_{L^2} \leq C$. Let $\{w_t\}_{t\in[-T,T]}$ be a limit of $W_\hbar\psi_\hbar(t)$ in $L^\infty([-T,+T];A')$ along a sequence of values of $\hbar\to 0$. Then,*

$$\partial_t w_t + \eta\cdot\nabla_x w_t - \nabla_x V(x)\cdot\nabla_\eta w_t = 0 \tag{4.96}$$ eq-Li1

in the distributional sense.

Proof. To begin, we prove that

$$\frac{d}{dt}\int_{\mathbb{T}^n\times\mathbb{R}^n}\phi(x,\eta)dw_t(x,\eta)+\int_{\mathbb{T}^n\times\mathbb{R}^n}\{\phi,H\}(x,\eta)dw_t(x,\eta)=0 \tag{4.97} \qquad \boxed{\texttt{eq-Liou}}$$

for any $\phi\in A$, see (2.33). To this aim, we observe that the Schrödinger equation implies

$$\frac{d}{dt}\langle\psi_\hbar(t),\mathrm{Op}^w_\hbar(\phi)\psi_\hbar(t)\rangle_{L^2}=(i\hbar)^{-1}\langle\psi_\hbar(t),[\mathrm{Op}^w_\hbar(H),\mathrm{Op}^w_\hbar(\phi)]\psi_\hbar(t)\rangle_{L^2}. \tag{4.98} \qquad \boxed{\texttt{eq-Hg}}$$

Hence

$$\langle\psi_\hbar(t),\mathrm{Op}^w_\hbar(\phi)\psi_\hbar(t)\rangle_{L^2}-\langle\varphi_\hbar,\mathrm{Op}^w_\hbar(\phi)\varphi_\hbar\rangle_{L^2}=\int_0^t(i\hbar)^{-1}\langle\psi_\hbar(s),[\mathrm{Op}^w_\hbar(H),\mathrm{Op}^w_\hbar(\phi)]\psi_\hbar(s)\rangle_{L^2}ds. \tag{4.99} \qquad \boxed{\texttt{eq-Hg22}}$$

where $\psi_\hbar(t=0)=:\varphi_\hbar\in L^2(\mathbb{T}^n;\mathbb{C})$ with $\|\varphi_\hbar\|_{L^2}\le C\ \forall 0<\hbar\le 1$. Morever, thanks to Theorem 2.4, the Weyl simbol of the commutator (namely the Moyal bracket of simbols H and ϕ) reads

$$\{H,\phi\}_{\mathrm{M}}=i\hbar\{H,\phi\}+r \tag{4.100}$$

where r has order $O(\hbar^2)$ when estimated in $S^{2+m}(\mathbb{T}^n\times\mathbb{R}^n)$ for any $m\in\mathbb{R}$, and thus also in $S^0(\mathbb{T}^n\times\mathbb{R}^n)$,

$$|\partial_x^\beta\partial_\eta^\alpha r(x,\eta)|\le C_{\alpha\beta}\,\hbar^2\langle\eta\rangle^{-|\alpha|}. \tag{4.101}$$

The related remainder operator $\mathrm{Op}^w_\hbar(r)$ is thus L^2-bounded, with (time independent) norm estimate thanks to Theorem 2.3 with order $O(\hbar^2)$. This directly gives

$$\lim_{\hbar\to0^+}\hbar^{-1}\Big|\int_0^t\langle\psi_\hbar(s),\mathrm{Op}^w_\hbar(r)\psi_\hbar(s)\rangle_{L^2}ds\Big|\le\lim_{\hbar\to0^+}t\,\hbar^{-1}\|\mathrm{Op}^w_\hbar(r)\|_{L^2\to L^2}=0, \tag{4.102}$$

since $\|\psi_\hbar(s)\|_{L^2}=\|\psi_\hbar(s=0)\|_{L^2}=\|\varphi_\hbar\|_{L^2}\le C$. The first term in (4.99) reads

$$\langle\psi_\hbar(t),\mathrm{Op}^w_\hbar(\phi)\psi_\hbar(t)\rangle_{L^2}=\sum_{\eta\in\frac{\hbar}{2}\mathbb{Z}^n}\int_{\mathbb{T}^n}\phi(x,\eta)W_\hbar\psi_\hbar(t,x,\eta)dx. \tag{4.103} \qquad \boxed{\texttt{348}}$$

Let $w_t(x,\eta)$ be a family of Radon measures of finite mass on $\mathbb{T}^n\times\mathbb{R}^n$ for any $t\in[-T,T]$ which is a limit of $W_\hbar\psi_\hbar$ in $L^\infty([-T,+T];A')$ along a sequence of values of $\hbar\to0$. The related semiclassical limit of (4.103) reads

$$\int_{\mathbb{T}^n\times\mathbb{R}^n}\phi(x,\eta)dw_t(x,\eta). \tag{4.104}$$

If we now look at

$$\sum_{\eta\in\frac{\hbar}{2}\mathbb{Z}^n}\int_{\mathbb{T}^n}\{H,\phi\}(x,\eta)W_\hbar\psi_\hbar(t,x,\eta)dx \tag{4.105}$$

we recall that ϕ is rapidly decreasing in η-variables and the phase space transform $\widehat{\phi}$ has compact support, hence also $\{H,\phi\}\in A$. As a consequence, we can extract a subsequence of the above one so that the semiclassical limit of the righthand side of (4.99) reads

$$\int_0^t\int_{\mathbb{T}^n\times\mathbb{R}^n}\{H,\phi\}(x,\eta)dw_s(x,\eta)ds. \tag{4.106}$$

We therefore deduce that $\forall t\in\mathbb{R}$

$$\int_{\mathbb{T}^n\times\mathbb{R}^n}\phi(x,\eta)dw_t(x,\eta)-\int_{\mathbb{T}^n\times\mathbb{R}^n}\phi(x,\eta)dw_0(x,\eta)=\int_0^t\int_{\mathbb{T}^n\times\mathbb{R}^n}\{H,\phi\}(x,\eta)dw_s(x,\eta)ds, \tag{4.107}$$

and observe that the righthand side is differentiable for any $t \in \mathbb{R}$ (and thanks to the equivalence, the lefthand side too). We now take the time derivative of both sides and get equation (4.97). On the other hand, since H is smooth, it is easily seen that equation (4.97) has a unique solution, and it is given by the push forward of the initial data $w_t = (\varphi_H^t)_\star(w_0)$ involving the Hamiltonian flow. However, this is also the unique solution of the Liouville equation written in the following weak sense

$$\int_0^t \int_{\mathbb{T}^n \times \mathbb{R}^n} [\partial_t f(t,x,\eta) + \{f,H\}(t,x,\eta)]\, dw_s(x,\eta) ds = 0 \quad \forall f \in C_0^\infty([0,t] \times \mathbb{T}^n \times \mathbb{R}^n), \tag{4.108}$$ eq-Li22

see for example A [1]. □

SEC-wkb
4.2 WKB wave functions of positive and negative type

We begin this section introducing a class of WKB-type wave functions in $H^1(\mathbb{T}^n;\mathbb{C})$ associated with weak KAM solutions of the stationary Hamilton-Jacobi equation.

wave-d **Definition 4.2.** *Let $P \in \ell\mathbb{Z}^n$ for some $\ell > 0$ and $\hbar^{-1} \in \ell^{-1}\mathbb{N}$. Let $v_\pm(P,\cdot) \in C^{0,1}(\mathbb{T}^n;\mathbb{R})$ be weak KAM solutions of the H-J equation (2.57) (in the sense of* F *[13], see subsection 2.2.1). Select $a^\pm_{\hbar,P} \in H^1(\mathbb{T}^n;\mathbb{R}^+)$ such that*

$$\mathrm{dom}(a^\pm_{\hbar,P}) \subseteq \mathrm{dom}(\nabla_x v_\pm(P,\cdot)) \tag{4.109}$$ def-inc

$\|a^\pm_{\hbar,P}\|_{L^2} = 1$ and $\hbar\|a^\pm_{\hbar,P}\|_{H^1} \longrightarrow 0$ as $\hbar \longrightarrow 0^+$. We suppose that the following weak limit upon passing through a subsequence $\hbar_j \longrightarrow 0^+$

$$\exists\ dm^\pm_P(x) := \lim_{\hbar_j \longrightarrow 0^+} |a^\pm_{\hbar_j,P}(x)|^2 dx \tag{4.110}$$ def-dm

fulfills $dm_P \ll \pi_\star(dw_P) =: d\sigma_P$ where dw_P is a Mather P-minimal measure as in (2.66). The WKB wave functions of negative type are defined by

$$\varphi^-_\hbar(x) := a^-_{\hbar,P}(x)\ e^{\frac{i}{\hbar}[P\cdot x + v_-(P,x)]}. \tag{4.111}$$ wf-d

The WKB wave functions of positive type are given by

$$\varphi^+_\hbar(x) := a^+_{\hbar,P}(x)\ e^{\frac{i}{\hbar}[P\cdot x + v_+(P,x)]}. \tag{4.112}$$ wf-dp

EX-a **Remark 4.3** (**Example**). *About the previous definition, we exhibit an explicit construction for $a^\pm_{\hbar,P}$. In fact, consider $\rho \in C_0^\infty(\mathbb{R}^n)$ such that $0 \le \rho$, $\mathrm{supp}\,\rho \subset Q_n := [0,2\pi]^n$ and $\int \rho(x)dx = 1$. For a fixed $\alpha > 0$ let*

$$\Phi_{\alpha,\hbar}(x) := \hbar^{-n\alpha} \sum_{k\in\mathbb{Z}^n} \rho\Big(\frac{x-2\pi k}{\hbar^\alpha}\Big). \tag{4.113}$$ eqMillifier

Then $\int_{\mathbb{T}^n} \Phi_{\alpha,\hbar}(x)dx = 1$, and if $f \in L^1(\mathbb{T}^n)$ we have, by the periodicity,

$$\Phi_{\alpha,\hbar} \star f(x) = \int_{\mathbb{T}^n} \Phi_{\alpha,\hbar}(x-y) f(y) dy = \int_{Q_n} \rho(z) f(x - \hbar^\alpha z) dz$$

Fix a fixed (P-dependent) Borel positive measure $dm^\pm_P$ on $\mathbb{T}^n$ with $\mathrm{supp}(dm^\pm_P) \subseteq \mathrm{dom}(\nabla_x v_\pm(P,\cdot))$, an amplitude function can be given by

$$a^\pm_{\hbar,P}(x) := \Big\{ \int_{\mathbb{T}^n} \frac{1}{c_0}\Big(\hbar^\epsilon + \Phi_{\gamma,\hbar}(x-y)\Big) dm_P(y) \Big\}^{1/2}\Big|_{\mathrm{dom}(\nabla v_\pm)}, \tag{4.114}$$ ampl

where $\epsilon,\gamma > 0$ with $0 < \epsilon + \gamma(n+1) < 1$, $c_0 = c_0(\hbar) = \|\hbar^\epsilon + \rho\|_{L^1(Q_n)} = 1 + O(\hbar^\epsilon)$. Notice that $a > \hbar^{\epsilon/2} c_0^{-1/2}$ then since $x \mapsto a^\pm_{\hbar,P}(x)$ is 2π-periodic (in each variable), it is a well-defined function on the torus. The function (4.114) is in $C^k(\mathbb{T}^n;\mathbb{R}^+)$, $\forall\ k \in \mathbb{N}$, and fulfills (see Prop 4.5 in P-Z *[23])*

(i) $\displaystyle\int_{\mathbb{T}^n} |a^{\pm}_{\hbar,P}(x)|^2 dx = 1;$

(ii) $\displaystyle\hbar^2 \int_{\mathbb{T}^n} |\nabla_x a^{\pm}_{\hbar,P}(x)|^2 dx \leq \|\nabla_x \rho\|^2_{L^\infty}\, \hbar^{2(1-\epsilon-(n+1)\gamma)};$

(iii) $\displaystyle\lim_{\hbar\to 0+} \int_{\mathbb{T}^n} f(x)|a^{\pm}_{\hbar,P}(x)|^2 dx = \int_{\mathbb{T}^n} f(x) dm^{\pm}_P(x), \quad \forall f \in C^0(\mathbb{T}^n;\mathbb{R}),$

(iv) $\displaystyle\lim_{\hbar\to 0+} \int_{\mathbb{T}^n} f(x)|a^{\pm}_{\hbar,P}(x)|^2 dx = \int_{\mathbb{T}^n} f(x) dm^{\pm}_P(x),$ $\forall$ *bounded Borel measurable* $f\colon \mathbb{T}^n \longrightarrow \mathbb{R}$ *whose discontinuity set has zero* dm_P*-measure.*

In the following, we provide two useful Lemma involving our class of WKB functions.

Lemma 4.4. *Let* $\varphi^{\pm}_{\hbar}$ *be as in Definition 4.2. Then,* $\varphi^{\pm}_{\hbar} \in H^1(\mathbb{T}^n;\mathbb{C})$.

Proof. The L^2-norm simply reads $\|\varphi^{\pm}_{\hbar}\|_{L^2} = \|a^{\pm}_{\hbar,P}\|_{L^2} < +\infty$, whereas

$$\|\nabla_x \varphi^{\pm}_{\hbar}\|_{L^2} \leq \frac{1}{\hbar}\|(P+\nabla_x v_{\pm})a^{\pm}_{\hbar,P}\|_{L^2} + \|\nabla_x a^{\pm}_{\hbar,P}\|_{L^2}$$

Recalling (2.61) and the setting of $a_{\hbar,P}$, it follows

$$\|\nabla_x \varphi^{\pm}_{\hbar}\|_{L^2} \leq \frac{1}{\hbar}\|P+\nabla_x v_{\pm}\|_{L^\infty} + \|a^{\pm}_{\hbar,P}\|_{H^1} < +\infty \quad \forall\, 0<\hbar\leq 1.$$

□

L-T1 **Lemma 4.5.** *Let* $\varphi^{\pm}_{\hbar}$ *be as in Definition 4.2. Let* $\mathbb{P}^{\pm}_{\hbar}$ *be as in (2.41) associated to* $\varphi^{\pm}_{\hbar}$*. Then, the family of measures* $\{\mathbb{P}^{\pm}_{\hbar}\}_{0\leq\hbar\leq 1}$ *is tight.*

Proof. Let $M_R := \mathbb{T}^n \times \{\mathbb{R}^n \backslash B_R\}$ and $U_R := \mathbb{R}^n \backslash B_R$. Thanks to (2.52)

$$\mathbb{P}^{\pm}_{\hbar}(\mathbb{T}^n \times U_R) = \sum_{\alpha\in\mathbb{Z}^n} \mathcal{X}_{U_R}(\hbar\alpha)|\widehat{\phi}^{\pm}_{\hbar,\alpha}|^2 \tag{4.115}$$

where the Fourier components read

$$\begin{aligned}\widehat{\phi}^{\pm}_{\hbar,\alpha} &:= (2\pi)^{-n}\int_{\mathbb{T}^n} e^{-i\alpha\cdot x}\varphi^{\pm}_{\hbar}(x)dx = (2\pi)^{-n}\int_{\mathbb{T}^n} e^{-i\alpha\cdot x} a^{\pm}_{\hbar,P}(x)\; e^{\frac{i}{\hbar}[P\cdot x + v_{\pm}(P,x)]}dx && (4.116)\\ &= (2\pi)^{-n}\int_{\mathbb{T}^n} a^{\pm}_{\hbar,P}(x)\; e^{\frac{i}{\hbar}v_{\pm}(P,x)} e^{\frac{i}{\hbar}(-\hbar\alpha+P)\cdot x}dx && (4.117)\end{aligned}$$

and $P \in \ell\mathbb{Z}^n$ for some fixed $\ell > 0$; moreover we underline thar the series (4.115) is computed over $|\hbar\alpha| > R$ (or equivalently $|\alpha| > R\hbar^{-1}$). In the case $R > |P|$, it holds the equality

$$\widehat{\phi}^{\pm}_{\hbar,\alpha} = \frac{(-i\hbar)}{|-\hbar\alpha+P|^2}(-\hbar\alpha+P)\cdot(2\pi)^{-n}\int_{\mathbb{T}^n} a^{\pm}_{\hbar,P}(x)\; e^{\frac{i}{\hbar}v_{\pm}(P,x)}\nabla_x e^{\frac{i}{\hbar}(-\hbar\alpha+P)\cdot x}dx. \tag{4.118}$$

The integration by parts gives

$$\begin{aligned}\widehat{\phi}^{\pm}_{\hbar,\alpha} &= \frac{(i\hbar)}{|-\hbar\alpha+P|^2}(-\hbar\alpha+P)\cdot(2\pi)^{-n}\int_{\mathbb{T}^n} \nabla_x a^{\pm}_{\hbar,P}(x)\; e^{\frac{i}{\hbar}v_{\pm}(P,x)} e^{\frac{i}{\hbar}(-\hbar\alpha+P)\cdot x}dx && (4.119)\\ &- \frac{1}{|-\hbar\alpha+P|^2}(-\hbar\alpha+P)\cdot(2\pi)^{-n}\int_{\mathbb{T}^n} a^{\pm}_{\hbar,P}(x)(\nabla_x v_{\pm}(P,x)) e^{\frac{i}{\hbar}v_{\pm}(P,x)} e^{\frac{i}{\hbar}(-\hbar\alpha+P)\cdot x}dx\end{aligned}$$

We are now in the position to provide an estimate for $|\widehat{\phi}^{\pm}_{\hbar,\alpha}|$, indeed some easy computations together with the application of Cauchy-Schwarz inequality give

$$|\widehat{\phi}^{\pm}_{\hbar,\alpha}| \quad \leq \quad \frac{(2\pi)^{-n/2}}{|-\hbar\alpha+P|}\Big(\|\hbar\nabla_x a^{\pm}_{\hbar,P}\|_{L^2}+\|\nabla_x v_{\pm}(P,\cdot)\|_{L^\infty}\Big) \tag{4.120}$$

Recalling (2.61) we have $\|\nabla_x v_{\pm}(P,\cdot)\|_{L^\infty} < +\infty$ for any fixed $P \in \ell\mathbb{Z}^n$. We also remind that $\|\hbar\nabla_x a^{\pm}_{\hbar,P}\|_{L^2} \to 0$ as $\hbar \to 0^+$. To conclude, by defining

$$C_{n,P} := (2\pi)^{-n}\Big(\sup_{0<\hbar\leq 1}(\|\hbar\nabla_x a^{\pm}_{\hbar,P}\|_{L^2})+\|\nabla_x v_{\pm}(P,\cdot)\|_{L^\infty}\Big)^2 \tag{4.121}$$

it follows (when $R > |P|$)

$$|\mathbb{P}^{\pm}_{\hbar}(\mathbb{T}^n \times U_R)| \leq \sum_{\alpha\in\mathbb{Z}^n, |\hbar\alpha|>R} \frac{C_{n,P}}{|-\hbar\alpha+P|^2} \leq \int_{\mathbb{R}^n/B_R(0)} \frac{C_{n,P}}{|-y+P|^2}dy \tag{4.122}$$

The last ($\hbar$-independent) upper bound implies that

$$\lim_{R\to+\infty}\sup_{0<\hbar\leq 1}|\mathbb{P}^{\pm}_{\hbar}(\mathbb{T}^n \times U_R)| = 0. \tag{4.123}$$

□

We next exhibit a property of the involved monokinetic measures.

MM **Proposition 4.6.** *Let $dm^{\pm}_P$ as in (4.110) and $v_-(P,\cdot) \in C^{0,1}(\mathbb{T}^n;\mathbb{R})$ be a weak KAM solution of negative type for the H-J equation (2.57). Define the lifted Borel measure on $\mathbb{T}^n \times \mathbb{R}^n$ by*

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} \phi(x,\eta)d\widetilde{m}^{\pm}_P(x,\eta) := \int_{\mathbb{T}^n\times\mathbb{R}^n} \phi(x,P+\nabla_x v_-(P,x))dm^{\pm}_P(x), \quad \forall\ \phi \in A. \tag{4.124}$$ mono

Then, $d\widetilde{m}^{\pm}_P$ does not depend on the choice of $v_-(P,\cdot)$, namely

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} \phi(x,\eta)d\widetilde{m}^{\pm}_P(x,\eta) = \int_{\mathbb{T}^n} \phi(x,P+\nabla_x v'_-(P,x))dm^{\pm}_P(x) \tag{4.125}$$

for any other weak KAM of negative type $v'_-(P,x)$. Moreover, for any weak KAM of positve type $v_+(P,x)$ it holds

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} \phi(x,\eta)d\widetilde{m}^{\pm}_P(x,\eta) = \int_{\mathbb{T}^n} \phi(x,P+\nabla_x v_+(P,x))dm^{\pm}_P(x) \tag{4.126}$$

Finally, there exists a Borel measurable function $g^{\pm}(P,\cdot) : \mathbb{T}^n \to \mathbb{R}^+$ such that

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} \phi(x)dm^{\pm}_P(x) = \int_{\mathbb{T}^n} \phi(x)\ g^{\pm}(P,x)d\sigma^{\pm}_P(x). \tag{4.127}$$ abs

Proof. For any $v_{\pm}(P,\cdot) \in C^{0,1}(\mathbb{T}^n;\mathbb{R})$ which is a weak KAM solution of Hamilton-Jacobi equation (2.57), the map $x \mapsto \nabla_x v_{\pm}(P,x)$ is continuous and uniformely bounded on its domain of definition $\mathrm{dom}(\nabla_x v_{\pm}(P,\cdot)) \subseteq \mathbb{T}^n$. Moreover, since we assumed $dm^{\pm}_P \ll d\sigma_P$ then $\mathrm{supp}(dm^{\pm}_P) \subseteq \mathrm{supp}(d\sigma_P)$. By recalling that $\mathrm{supp}(d\sigma_P) \subseteq \pi(\mathcal{M}^{\star}_P) \subseteq \pi(\mathcal{A}^{\star}_P)$ and thanks to the localization the Aubry set $\mathcal{A}^{\star}_P$ shown in Section 2.2.3, it follows

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} \phi(x,\eta)d\widetilde{m}^{\pm}_P(x,\eta) = \int_{\mathbb{T}^n} \phi(x,P+\nabla_x v_{\pm}(P,x))dm^{\pm}_P(x) \tag{4.128}$$

for any $v_{\pm}(P,\cdot) \in C^{0,1}(\mathbb{T}^n;\mathbb{R})$ weak KAM solutions of Hamilton-Jacobi equation. Finally, the assumption on the absolute continuity of $dm^{\pm}_P$ with respect to $d\sigma_P$ together with the well known Radon-Nikodym derivative provides the existence of $g^{\pm}(P,x)$ satisfying (4.127). □

Lem-ac **Lemma 4.7.** *Let*

$$d\widetilde{m}^{\pm}_P(x,\eta) := \delta(\eta - P - \nabla_x v_{\pm}(P,x))dm^{\pm}_P(x) \tag{4.129}$$ mP

be as in Proposition 4.6. Then, $d\widetilde{m}^{\pm}_P$ is absolutely continuous to dw_P the Legendre transform of a Mather P-minimal measure. In particular, there exists a Borel measurable function $g^{\pm}(P,\cdot) : \mathbb{T}^n \to \mathbb{R}^+$ such that

$$d\widetilde{m}^{\pm}_P(x,\eta) = g^{\pm}(P,x)dw_P(x,\eta) \tag{4.130}$$ mP2

where $dw_P(x,\eta) = \delta(\eta - P - \nabla_x v_{\pm}(P,x))d\sigma_P(x)$.

Proof. By the assumption within Definition 4.2, it holds $dm_P \ll \pi_\star(dw_P) =: d\sigma_P$ where dw_P is Legendre transform of a Mather P-minimal measure $d\mu_P$ as in (2.66). Equivalently, we can also take $\pi_\star(d\mu_P) =: d\sigma_P$ since the push forward by the canonical projection is the same. Thus, there exists a Borel measurable function $g^{\pm}(P,\cdot) : \mathbb{T}^n \to \mathbb{R}^+$ such that

$$d\widetilde{m}^{\pm}_P(x,\eta) = g^{\pm}(P,x)\delta(\eta - P - \nabla_x v_{\pm}(P,x))d\sigma_P(x). \tag{4.131}$$

In fact, it holds the equality $\delta(\eta - P - \nabla_x v_{\pm}(P,x))d\sigma_P(x) = dw_P(x,\eta)$ thanks to the inclusion

$$\mathrm{supp}(dw_P) \subseteq \mathcal{A}^\star_P \subseteq \mathrm{Graph}(P + \nabla_x v_{\pm}(P,\cdot)),$$

see Lemma 3.1 shown in [14] (F-G-S). The (4.130) follows directly. □

We are now ready to provide the result involving the semiclassical limits of the Wigner transform for the above class of WKB-type wave functions.

TH4 **Theorem 4.8.** *Let $P \in \ell\,\mathbb{Z}^n$ for some $\ell > 0$, $\hbar^{-1} \in \ell^{-1}\mathbb{N}$, $v_\pm$ be weak KAM solutions of H-J equation (2.57) and $\varphi^{\pm}_\hbar$ be the associated WKB wave functions as in Def. 4.2, $dm^{\pm}_P$ as in Def. 4.2. Then,*

$$\lim_{\hbar\to 0^+} W_\hbar\varphi^{\pm}_\hbar(x,\eta) = \delta(\eta - P - \nabla_x v_{\pm}(P,x))dm^{\pm}_P(x) =: d\widetilde{m}^{\pm}_P(x,\eta) \tag{4.132}$$ WKB-c

in A' and passing through a subsequence.

Proof. The Wigner transform in the variables $(q,p) \in \mathbb{Z}^n \times \mathbb{R}^n$:

$$\begin{aligned}\widehat{W}_\hbar\varphi^{\pm}_\hbar(q,p) &:= \int_{\mathbb{T}^n} \varphi^{\pm}_\hbar(y)^\star e^{i(q\cdot y + \hbar p\cdot q/2)}\varphi^{\pm}_\hbar(y+\hbar p)dy \\ &= \int_{\mathbb{T}^n} e^{i[\hbar p\cdot q/2 + P\cdot p]}\ e^{iq\cdot y} e^{\frac{i}{\hbar}[v_\pm(P,y+\hbar p) - v_\pm(P,y)]} a^{\pm}_{\hbar,P}(y)a^{\pm}_{\hbar,P}(y+\hbar p)dy. \end{aligned}\tag{4.133}$$

Since $a^{\pm}_{\hbar,P} \in L^2(\mathbb{T}^n;\mathbb{R}^+)$, the integral in (4.133) is absolutely convergent and the function $\widehat{W}_\hbar\varphi^{\pm}_\hbar(\cdot)$ is Lebesgue measurable and uniformely bounded in both variables.
By the H^1-regularity we can write $a^{\pm}_{\hbar,P}(y+\hbar p) = a^{\pm}_{\hbar,P}(y) + \hbar\int_0^1 p\cdot\nabla_x a^{\pm}_{\hbar,P}(y+\lambda\hbar p)d\lambda$ and hence

$$\|a^{\pm}_{\hbar,P}(\diamond + \hbar p) - a^{\pm}_{\hbar,P}(\diamond)\|_{L^2} \le |p|\ \hbar\ \|a^{\pm}_{\hbar,P}\|_{H^1} \tag{4.134}$$

Thus,

$$\widehat{W}_\hbar\varphi^{\pm}_\hbar(q,p) = \int_{\mathbb{T}^n} e^{i[\hbar p\cdot q/2 + P\cdot p]}\ e^{iq\cdot y} e^{\frac{i}{\hbar}[v_\pm(P,y+\hbar p) - v_\pm(P,y)]} a^{\pm}_{\hbar,P}(y)^2 dy + R_\hbar(q,p) \tag{4.135}$$

where

$$R_\hbar(q,p) := \int_{\mathbb{T}^n} e^{i[\hbar p\cdot q/2 + P\cdot p]}\ e^{iq\cdot y} e^{\frac{i}{\hbar}[v_\pm(P,y+\hbar p) - v_\pm(P,y)]} a^{\pm}_{\hbar,P}(y)[a^{\pm}_{\hbar,P}(y+\hbar p) - a^{\pm}_{\hbar,P}(y)]dy$$

and thus $\forall (q,p) \in \mathbb{Z}^n \times \mathbb{R}^n$

$$|R_\hbar(q,p)| \le \mathrm{vol}(\mathbb{T}^n)\|a_{\hbar,P}\|_{L^2}\|a_{\hbar,P}(\diamond + \hbar p) - a_{\hbar,P}(\diamond)\|_{L^2} \le (2\pi)^n\ |p|\ \hbar\ \|a_{\hbar,P}\|_{H^1}. \tag{4.136}$$

For any $\phi \in A$ and $\mathrm{supp}(\phi)$ is compact,

$$\sum_{q\in\mathbb{Z}^n}\int_{\mathbb{R}^n}\widehat{\phi}(q,p)\widehat{W}_\hbar\varphi_\hbar^\pm(q,p)(q,p)dp \tag{4.137}$$

$$= \sum_{q\in\mathbb{Z}^n}\int_{\mathbb{R}^n}\widehat{\phi}(q,p)\int_{\mathbb{T}^n} e^{i[\hbar p\cdot q/2+P\cdot p]}\ e^{iq\cdot y}e^{\frac{i}{\hbar}[v_\pm(P,y+\hbar p)-v_\pm(P,y)]}a_{\hbar,P}(y)^2dydp$$

$$+ \sum_{q\in\mathbb{Z}^n}\int_{\mathbb{R}^n}\widehat{\phi}(q,p)R_\hbar(q,p)dp. \tag{4.138}$$

An easy computation shows that

$$\Big|\sum_{q\in\mathbb{Z}^n}\int_{\mathbb{R}^n}\widehat{\phi}(q,p)R_\hbar(q,p)dp\Big| \le \sum_{q\in\mathbb{Z}^n}\int_{\mathbb{R}^n}|\widehat{\phi}(q,p)|(2\pi)^n\ |p|\ \hbar\ \|a_{\hbar,P}\|_{H^1}dp$$

and hence, since $\mathrm{supp}(\widehat{\phi})$ is compact and $\hbar\|a^\pm_{\hbar,P}\|_{H^1}\longrightarrow 0$ as $\hbar\longrightarrow 0^+$ (see Remark 4.3) it follows

$$(2\pi)^n\sum_{q\in\mathbb{Z}^n}\int_{\mathbb{R}^n}|\widehat{\phi}(q,p)|\ |p|dp\ \hbar\ \|a^\pm_{\hbar,P}\|_{H^1}\longrightarrow 0^+ \quad \text{as} \quad \hbar\longrightarrow 0^+. \tag{4.139}$$ est-R2

In view of (4.139) and the compactness of $\mathrm{supp}(\widehat{\phi})$, the (4.137) reads

$$\sum_{q\in\mathbb{Z}^n}\int_{\mathbb{R}^n}\widehat{\phi}(q,p)\lim_{\hbar\to 0^+}\int_{\mathbb{T}^n} e^{i[\hbar p\cdot q/2+P\cdot p]}\ e^{iq\cdot y}e^{\frac{i}{\hbar}[v_\pm(P,y+\hbar p)-v_\pm(P,y)]}|a^\pm_{\hbar,P}(y)|^2dydp.$$

By looking at the integral

$$\int_{\mathbb{T}^n} e^{i[\hbar p\cdot q/2+P\cdot p]}\ e^{iq\cdot y}e^{\frac{i}{\hbar}[v_\pm(P,y+\hbar p)-v_\pm(P,y)]}|a^\pm_{\hbar,P}(y)|^2dy \tag{4.140}$$ 410

we observe that $e^{i(\hbar p\cdot q/2)}e^{\frac{i}{\hbar}[v_\pm(P,y+\hbar p)-v_\pm(P,y)]}$ is a family of uniformely bounded continuous functions on $\mathbb{T}^n$ such that

$$\lim_{\hbar\to 0^+} e^{i(\hbar p\cdot q/2)}e^{\frac{i}{\hbar}[v_\pm(P,y+\hbar p)-v_\pm(P,y)]} = e^{ip\cdot\nabla_x v_\pm(P,y)} \tag{4.141}$$ 412

$\forall (q,p)\in\mathrm{supp}(\widehat{\phi})$ and $\forall\ y\in\mathrm{dom}(\nabla_x v_\pm(P,\cdot))$, since any map $x\longmapsto\nabla_x v_\pm(P,x)$ is continuous on $\mathrm{dom}(\nabla_x v_\pm(P,\cdot))$ (as we recall in Section 2.2.1). By the inclusions

$$\mathrm{supp}(dm_P^\pm)\subseteq\mathrm{supp}(d\sigma_P)\subseteq\mathrm{dom}(\nabla_x v_\pm(P,\cdot)) \tag{4.142}$$

we deduce that (4.141) is not fulfilled only for a set of zero $dm_P^\pm$ measure.
Hence, we can apply Lemma 6.4 for the semiclassical limits of the integral (4.140) to obtain

$$\int_{\mathbb{T}^n} e^{iP\cdot p}\ e^{iq\cdot y}e^{ip\cdot\nabla_x v_\pm(P,y)}dm_P^\pm(y)dp. \tag{4.143}$$

We deduce that the semiclassical limits of the mean value (4.140) read

$$\sum_{q\in\mathbb{Z}^n}\int_{\mathbb{R}^n}\widehat{\phi}(q,p)\Big(\int_{\mathbb{T}^n} e^{iP\cdot p}\ e^{iq\cdot y}e^{ip\cdot\nabla_x v_\pm(P,y)}dm_P^\pm(y)\Big)dp. \tag{4.144}$$

$$= \int_{\mathbb{T}^n}\sum_{q\in\mathbb{Z}^n}\int_{\mathbb{R}^n}\widehat{\phi}(q,p)e^{iP\cdot p}\ e^{iq\cdot y}e^{ip\cdot\nabla_x v_\pm(P,y)}dp\ dm_P^\pm(y) \tag{4.145}$$

where we used again the compacteness of $\text{supp}(\widehat{\phi})$. Through the inverse phase-space Fourier transform the above expression becomes

$$\int_{\mathbb{T}^n} \phi(y, P + \nabla_x v_\pm(P, y))\, dm_P^\pm(y). \tag{4.146}$$

$\square$

Remark 4.9. *Let $P \in \ell\,\mathbb{Z}^n$ for some $\ell > 0$ and $\varphi_\hbar^\pm$ as in Definition 4.2. Define the current*

$$J_\hbar^\pm(x) := \hbar\, \text{Im}((\varphi_\hbar^\pm)^\star \nabla_x \varphi_\hbar^\pm(x)) = (P + \nabla_x v_\pm(P, x))|a_{\hbar,P}^\pm(x)|^2 \tag{4.147}$$ def-J

The (formal) free current equation $\text{div}_x J_\hbar^\pm(x) = 0$ becomes well-posed in the weak sense:

$$\int_{\mathbb{T}^n} \nabla_x f(x) \cdot J_\hbar^\pm(x)\; dx = 0 \qquad \forall f \in C^\infty(\mathbb{T}^n; \mathbb{R}). \tag{4.148}$$

In particular, we recall the inclusion (2.61) which implies, together with the assumptions on $a_{\hbar,P}^\pm$, the estimate $\sup_{0<\hbar\le 1} \|J_\hbar^\pm\|_{L^1} \le \|P + \nabla_x v_\pm(P, \cdot)\|_{L^\infty} < +\infty$. However, the low regularity $v_\pm(P, \cdot) \in C^{0,1}(\mathbb{T}^n; \mathbb{R}^n)$ do not guarantees the existence of some amplitude function satifying this equation, hence we write the asymptotic condition

$$\Big| \int_{\mathbb{T}^n} \nabla_x f(x) \cdot J_{\hbar_j}^\pm(x)\; dx \Big| \longrightarrow 0, \quad \forall f \in C^\infty(\mathbb{T}^n; \mathbb{R}) \tag{4.149}$$

for a sequence $\{\hbar_j^{-1}\}_{j\in\mathbb{N}} \in \ell^{-1}\mathbb{N}$ with $\hbar_j \longrightarrow 0^+$ as $j \longrightarrow +\infty$.

The above observations become meaningful in view of the following result.

prop46 **Proposition 4.10.** *Let $P \in \ell\,\mathbb{Z}^n$ for some $\ell > 0$, $v_\pm(P, \cdot) \in C^{0,1}(\mathbb{T}^n; \mathbb{R})$ be a weak KAM solution for (2.57). Then, there exist $a_{\hbar,P}^\pm$ as in Remark 4.3 such that the (unique) weak-$\star$ limit $dm_P(x) := \lim_{j\to+\infty} |a_{\hbar_j,P}^\pm(x)|^2 dx$ equal $d\sigma_P := \pi_\star(dw_P)$ where dw_P is the Legendre transform of a Mather P-minimal measure and*

$$\Big| \int_{\mathbb{T}^n} \nabla_x f(x) \cdot J_{\hbar_j}^\pm(x) dx \Big| \longrightarrow 0 \quad \text{as } j \longrightarrow +\infty \quad \forall f \in C^\infty(\mathbb{T}^n; \mathbb{R}). \tag{4.150}$$ divJ1

Proof. Let $d\sigma_P := \pi_\star(dw_P) = d\mu_P$ with dw_P as in (2.66). Then, $d\sigma_P$ is a Borel probability measure $\mathbb{T}^n$ with

$$\text{supp}(d\sigma_P) \subseteq \pi_\star(\mathcal{M}_P^\star) \subseteq \pi_\star(\mathcal{A}_P^\star) \subseteq \text{dom}(\nabla_x v_\pm(P, \cdot)). \tag{4.151}$$ incl-s

Moreover, it holds

$$\int_{\mathbb{T}^n} \nabla_x f(x) \cdot (P + \nabla_x v_\pm(P, x))\; d\sigma_P(x) = 0 \quad \forall f \in C^\infty(\mathbb{T}^n; \mathbb{R}). \tag{4.152}$$ f-div

Indeed, $dw_P := \mathcal{L}_\star(d\mu_P)$ and $d\mu_P$ is invariant under Lagrangian flow, hence closed, which means that

$$\int_{\mathbb{T}^n \times \mathbb{R}^n} \nabla_x f(x) \cdot \xi\; d\mu_P(x, \xi) = 0 \quad \forall f \in C^\infty(\mathbb{T}^n; \mathbb{R}).$$

Here the Lagrangian reads $L(x, \xi) = \frac{1}{2}|\xi|^2 + V(x)$ and thus the Legendre transform $\mathcal{L}(x, \xi) = (x, \xi)$, which gives

$$\int_{\mathbb{T}^n \times \mathbb{R}^n} \nabla_x f(x) \cdot \eta\; dw_P(x, \eta) = 0 \quad \forall f \in C^\infty(\mathbb{T}^n; \mathbb{R}).$$

By Lemma 3.1 in F-G-S [14], we have necessary $\mathrm{supp}(dw_P) \subseteq \mathcal{A}^{\star}_P \subseteq \mathrm{Graph}(P + \nabla_x v_{\pm}(P,\cdot))$. Thus, we can restrict $dw_P|_{\mathrm{Graph}(P+\nabla_x v_{\pm}(P,\cdot))}$ since $\mathrm{Graph}(P + \nabla_x v_{\pm}(P,\cdot))$ are Borel measurable subsets of $\mathbb{T}^n \times \mathbb{R}^n$ containing the support of this measure. Hence

$$\int_{\mathrm{Graph}(P+\nabla_x v_{\pm}(P,\cdot))} \nabla_x f(x)\cdot\eta \; dw_P(x,\eta) = 0 \quad \forall f \in C^{\infty}(\mathbb{T}^n;\mathbb{R}).$$

The canonical projection $\pi : \mathrm{Graph}(P + \nabla_x v_{\pm}(P,\cdot)) \to \mathbb{T}^n$ is a Borel measurable map, because of $\overline{\mathrm{Graph}(P + \nabla_x v_{\pm}(P,\cdot))} = \mathbb{T}^n$. We can apply the change of variables and get (4.152).
Now, define the Borel probability measure $dm_P(x) := d\sigma_P(x)$ on $\mathbb{T}^n$. Recalling Remark 4.3, there exists $a^{\pm}_{\hbar,P} \in C^k(\mathbb{T}^n;\mathbb{R}^+)$ such that $\lim_{\hbar_j\to 0^+} |a^{\pm}_{\hbar_j,P}(x)|^2 = dm_P(x)$ in the weak-$\star$ convergence of Borel measures on $\mathbb{T}^n$. Notice that now we do not write dm_P as $dm^{\pm}_P$ since in fact holds the inclusion (4.151).
Thus, we look at

$$\int_{\mathbb{T}^n} \nabla_x f(x)\cdot J^{\pm}_{\hbar}(x)dx = \int_{\mathbb{T}^n} \nabla_x f(x)\cdot(P + \nabla_x v_{\pm}(P,x)) \; |a^{\pm}_{\hbar,P}(x)|^2 dx. \tag{4.153}$$

and observe that the function

$$x \longmapsto \nabla_x f(x)\cdot(P + \nabla_x v_{\pm}(P,x))$$

is a bounded Borel measurable function, and $x \mapsto \nabla_x v_{\pm}(P,x)$ is continuous on its domain of definition. Hence, the set of $x \in \mathbb{T}^n$ such that $\exists \, \{x_k\}_{k\in\mathbb{N}} \subset \mathbb{T}^n, \; \lim_{k\to+\infty} x_k = x$ and

$$\lim_{k\to+\infty} \nabla_x f(x_k)\cdot(P + \nabla_x v_{\pm}(P,x_k)) \;\neq \nabla_x f(x)\cdot(P + \nabla_x v_{\pm}(P,x))$$

is a set of zero dm_P-measure. We now apply Lemma 6.4 to get

$$\lim_{j\to+\infty} \int_{\mathbb{T}^n} \nabla_x f(x)\cdot(P + \nabla_x v_{\pm}(P,x)) \; |a^{\pm}_{\hbar_j,P}(x)|^2 dx \tag{4.154}$$
$$= \int_{\mathbb{T}^n} \nabla_x f(x)\cdot(P + \nabla_x v_{\pm}(P,x)) \; dm_P(x) = 0 \tag{4.155}$$

where the last equality is given by the above setting of $dm_P(x) := d\sigma_P(x)$ and (4.152). □

5 Propagation of Wigner measures on weak KAM tori

pwt

5.1 The forward and backward propagation

The main result of the section reads as

th51 **Theorem 5.1.** *Let $\varphi^{\pm}_{\hbar}$ be as in Def. 4.2 and $\psi_{\hbar}(t) := e^{-\frac{i}{\hbar}Op^w_{\hbar}(H)t}\varphi_{\hbar}$. Let $\widetilde{m}^{\pm}_P(t)$ be a limit of $W_{\hbar}\psi_{\hbar}(t)$ in $L^{\infty}([-T,+T];A')$, and $\widetilde{m}^{\pm}_P$, $g_{\pm}(P,x)$ be as in Proposition 4.6. Then, $\widetilde{m}^{\pm}_P(t) = (\varphi^t_H)_{\star}(\widetilde{m}^{\pm}_P) \in \mathcal{M}^{1+}(\mathbb{T}^n\times\mathbb{R}^n)$. Moreover, $\forall \phi \in A$ and $\forall \; t \geq 0$*

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} \phi(x,\eta) \; d\widetilde{m}^{+}_P(t,x,\eta) \;=\; \int_{\mathbb{T}^n} \phi(x, P + \nabla_x v_+(P,x)) \; \mathbf{g}_+(t,P,x) d\sigma_P(x) \tag{5.156}$$
$$\mathbf{g}_+(t,P,x) \;:=\; g_+(P, \pi\circ\varphi^{-t}_H(x, P + \nabla_x v_-(P,x))) \tag{5.157}$$

Whereas $\forall \; t \leq 0$

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} \phi(x,\eta) \; d\widetilde{m}^{-}_P(t,x,\eta) \;=\; \int_{\mathbb{T}^n} \phi(x, P + \nabla_x v_-(P,x)) \; \mathbf{g}_-(t,P,x) d\sigma_P(x) \tag{5.158}$$
$$\mathbf{g}_-(t,P,x) \;:=\; g_-(P, \pi\circ\varphi^{-t}_H(x, P + \nabla_x v_+(P,x))) \tag{5.159}$$

Proof. By Theorem 4.1, any limit $d\mathrm{w}(t)$ of the Wigner transform $W_\hbar\psi_\hbar(t)$ in $L^\infty([-T,+T];A')$ solves the Liouville equation in the distributional sense $L^\infty([-T,+T];A')$ and hence, thanks to the uniqueness for the solutions of this continuity equation, it holds $d\mathrm{w}(t) = (\varphi_H^t)_\star(d\mathrm{w}(0))$. This also implies that $d\mathrm{w}(\cdot) \in C([-T,+T];A')$. On the other hand, for our initial data $\varphi_\hbar^\pm$ we proved, within Theorem 4.8, that the Wigner transform $W_\hbar\varphi_\hbar^\pm$ is weak converging (for test functions in A) to the monokinetic probability measures $d\widetilde{m}_P^\pm \in \mathcal{M}^{1+}(\mathbb{T}^n \times \mathbb{R}^n)$. Moreover, recalling Lemma 4.5, the complex measures $\mathbb{P}_\hbar^\pm$ are tight and hence their time evolution $\mathbb{P}_\hbar^\pm(t)$ is tight as well (see Proposition 2.11). This implies that there exist semiclassical limits of $\mathbb{P}_\hbar^\pm(t)$ in the sense of (2.42), namely there exist weak limits of $W_\hbar\psi_\hbar(t)$ with respect to test functions in $C_b(\mathbb{T}^n \times \mathbb{R}^n) \supset A$ to some (a priori complex) Borel probability measures for any fixed t. In fact, this means that it must be $d\mathrm{w}(t) = (\varphi_H^t)_\star(d\mathrm{w}(0) = d\widetilde{m}_P^\pm) \in \mathcal{M}^{1+}(\mathbb{T}^n \times \mathbb{R}^n)$. From now on, we write $d\widetilde{m}_P^\pm(t) := (\varphi_H^t)_\star(d\widetilde{m}_P^\pm)$.
Next, we underline that $\forall \phi, \psi \in A$

$$\begin{aligned}\int_{\mathbb{T}^n\times\mathbb{R}^n} \phi(x,\eta)\; d\widetilde{m}_P^\pm(t,x,\eta) &= \int_{\mathbb{T}^n\times\mathbb{R}^n} \phi\circ\varphi_H^t(x,\eta)\; d\widetilde{m}_P^\pm(x,\eta) &(5.160)\\ \int_{\mathbb{T}^n\times\mathbb{R}^n} \psi(x,\eta)\; d\widetilde{m}_P^\pm(x,\eta) &= \int_{\mathbb{T}^n} \psi(x,P+\nabla_x v_\pm(P,x))\; g_\pm(P,x)d\sigma_P(x). &(5.161)\end{aligned}$$

Hence

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} \phi(x,\eta)\; d\widetilde{m}_P^\pm(t,x,\eta) = \int_{\mathbb{T}^n} \phi\circ\varphi_H^t(x,P+\nabla_x v_\pm(P,x))\; g_\pm(P,x)d\sigma_P(x). \tag{5.162}$$

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We now recall that $d\sigma_P := \pi_\star(dw_P)$ where dw_P is the Legendre transform of a Mather P-minimal measure, which takes the monokinetic form

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} \phi(x,\eta)\; dw_P(x,\eta) = \int_{\mathbb{T}^n} \phi(x,P+\nabla_x v_\pm(P,x))\; d\sigma_P(x) \tag{5.163}$$

form-dw

and dw_P is invariant under the Hamiltonian flow. This is a consequence of the Lemma 3.1 in F-G-S [14], which gives $\mathrm{supp}(dw_P) \subseteq \mathcal{A}_P^\star$ and thanks to the inclusion $\mathcal{A}_P^\star \subseteq \mathrm{Graph}(P+\nabla_x v_\pm(P,\cdot))$.
Hence, we can rewrite

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} \phi(x,\eta)\; d\widetilde{m}_P^\pm(t,x,\eta) = \int_{\mathbb{T}^n\times\mathbb{R}^n} \phi\circ\varphi_H^t(x,\eta)\; g_\pm(P,\pi(x,\eta))dw_P(x,\eta). \tag{5.164}$$

1fg

By the generalized change of variables,

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} \phi(x,\eta)\; d\widetilde{m}_P^\pm(t,x,\eta) = \int_{\mathbb{T}^n\times\mathbb{R}^n} \phi(x,\eta)\; g_\pm(P,\pi\circ\varphi_H^{-t}(x,\eta))(\varphi_H^{-t})_\star dw_P(x,\eta) \tag{5.165}$$

13fg

and thanks to the invariance of dw_P,

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} \phi(x,\eta)\; d\widetilde{m}_P^\pm(t,x,\eta) = \int_{\mathbb{T}^n\times\mathbb{R}^n} \phi(x,\eta)\; g_\pm(P,\pi\circ\varphi_H^{-t}(x,\eta))dw_P(x,\eta). \tag{5.166}$$

By (5.163)

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} \phi(x,\eta)\; d\widetilde{m}_P^\pm(t,x,\eta) = \int_{\mathbb{T}^n} \phi(x,P+\nabla_x v_\pm(P,x))\; g(P,\pi\circ\varphi_H^{-t}(x,P+\nabla_x v_\pm(P,x)))\; d\sigma_P(x). \tag{5.167}$$

rep-dwt

Thus, we can define

$$\mathbf{g}_+(t,P,x) := g_+(P,\pi\circ\varphi_H^{-t}(x,P+\nabla_x v_-(P,x))) \quad \text{for } t\geq 0 \tag{5.168}$$

g+1

and

$$\mathbf{g}_-(t,P,x) := g_-(P,\pi\circ\varphi_H^{-t}(x,P+\nabla_x v_+(P,x))) \quad \text{for } t\leq 0. \tag{5.169}$$

g-1

□

Remark 5.2. *We notice that the supports of the measures $d\widetilde{m}^{\pm}_P(t)$ are contained, for any $t \in \mathbb{R}$, in the Mather set $\mathcal{M}^\star_P \subseteq \mathcal{A}^\star_P$ in the phase space which is invariant under the Hamiltonian flow as well as $\mathcal{A}^\star_P$. Hence, these are also contained in any set* $\mathrm{Graph}(P + \nabla_x v_\pm(P,\cdot))$ *and this means that we could write several possible equivalent Borel measurable density functions $\mathbf{g}_\pm(t,P,x)$. However, within the next result we underline that the functions $\mathbf{g}_+$ solve a forward continuity equation with respect to the vector field $P + \nabla_x v_+(P,\cdot)$ and $\mathbf{g}_-$ solve a backward equation with respect to $P + \nabla_x v_-(P,\cdot)$.*

prop53 **Proposition 5.3.** *Let $\mathbf{g}_\pm$ and $d\sigma_P$ as in Theorem 5.1. Then, $\forall f \in C^\infty([0,t]\times\mathbb{T}^n;\mathbb{R})$*

$$\int_0^t \int_{\mathbb{T}^n} [\partial_s f(s,x) + \nabla_x f(s,x)\cdot(P + \nabla_x v_+(P,x))]\ \mathbf{g}_+(s,P,x) d\sigma_P(x) ds = 0 \quad \text{for} \quad t \geq 0 \tag{5.170}$$

redc+

and

$$\int_0^t \int_{\mathbb{T}^n} [\partial_s f(s,x) + \nabla_x f(s,x)\cdot(P + \nabla_x v_-(P,x))]\ \mathbf{g}_-(s,P,x) d\sigma_P(x) ds = 0 \quad \text{for} \quad t \leq 0 \tag{5.171}$$

redc-

Proof. We recall $\varphi^t_H|_{\mathcal{A}^\star_P} : \mathcal{A}^\star_P \to \mathcal{A}^\star_P$ is a one parameter group of homeomorphisms on the closed invariant graph $\mathcal{A}^\star_P$ on $\mathbb{T}^n$, hence

$$\mathbf{g}_+\ d\sigma_P = \pi_\star d\widetilde{m}_P(t) = \pi_\star(\varphi^t_H)_\star d\widetilde{m}_P(0) = \pi_\star\Big(\varphi^t_H|_{\mathcal{A}^\star_P}\Big)_\star d\widetilde{m}_P(0) = \Big(\pi(\varphi^t_H|_{\mathcal{A}^\star_P})\Big)_\star d\widetilde{m}_P(0) \tag{5.172}$$

The map $\pi(\varphi^t_H|_{\mathcal{A}^\star_P}) : \pi(\mathcal{A}^\star_P) \to \pi(\mathcal{A}^\star_P)$ is a one parameter group of homeomorphisms associated with the vector field

$$\mathrm{b}_\pm(x) := \frac{d}{dt}\pi(\varphi^t_H(x, P + \nabla_x v_\pm(P,x)))\Big|_{t=0} = \nabla_\eta H(x, P + \nabla_x v_\pm(P,x)) \tag{5.173}$$

defined for any $x \in \pi(\mathcal{A}^\star_P)$ but also in the bigger sets $\mathrm{dom}(\nabla_x v_\pm(P,\cdot))$ defined a.e. $x \in \mathbb{T}^n$. Here $H(x,\eta) = \frac{1}{2}|\eta|^2 + V(x)$ and thus $\nabla_\eta H(x,\eta) = \eta$. About the regularity, we have $\mathrm{b}_\pm \in L^\infty(\mathbb{T}^n;\mathbb{R}^n)$. Write down the ODE

$$\dot\gamma = \mathrm{b}_\pm(\gamma) \tag{5.174}$$

ODE

with $\gamma(0) = x \in \mathrm{dom}(\nabla_x v_\pm(P,\cdot))$ but remind the inclusions (see Section 2.2.3)

$$\varphi^t_H\Big(\mathrm{Graph}(P + \nabla_x v_+(P,\cdot))\Big) \subseteq \mathrm{Graph}(P + \nabla_x v_+(P,\cdot)) \quad \forall t \geq 0 \tag{5.175}$$

$$\varphi^t_H\Big(\mathrm{Graph}(P + \nabla_x v_-(P,\cdot))\Big) \subseteq \mathrm{Graph}(P + \nabla_x v_-(P,\cdot)) \quad \forall t \leq 0. \tag{5.176}$$

neg-g

Thus, even if we have the low regularity $\mathrm{b}_\pm \in L^\infty(\mathbb{T}^n;\mathbb{R}^n)$ and not (in general) in the larger $W^{1,\infty}(\mathbb{T}^n;\mathbb{R}^n)$, the equation (5.174) is well posed and solved for $t \geq 0$ and $\gamma(0) = x \in \mathrm{dom}(\nabla_x v_+(P,\cdot))$, or in the case $t \leq 0$ and $\gamma(0) = x \in \mathrm{dom}(\nabla_x v_-(P,\cdot))$. We are now in the position to apply the same proof of Proposition 2.1 in [1] and get the statement.
About the explicit representation of the density g_+ for $t \geq 0$,

$$\int_{\mathbb{T}^n}\phi(x)\mathbf{g}_+(t,P,x)d\sigma_P(x) = \int_{\mathbb{T}^n}\phi(\pi(\varphi^t_H|_{\mathcal{A}^\star_P})(x))g_+(P,x)d\sigma_P(x) = \int_{\mathbb{T}^n}\phi(x)g_+(P,\pi(\varphi^{-t}_H|_{\mathcal{A}^\star_P})(x))d\sigma_P(x)$$

since $d\sigma_P$ is invariant under $\pi(\varphi^{-t}_H|_{\mathcal{A}^\star_P})$. We are now looking at the Hamiltonian flow for negative times, and we recall $\mathrm{supp}(d\sigma_P) \subseteq \mathcal{M}^\star_P \subseteq \mathcal{A}^\star_P \subseteq \mathrm{Graph}(P + \nabla_x v_\pm(P,\cdot))$, thus we can choose the solution

$$\mathbf{g}_+(t,P,x) = g_+(P, \pi\circ\varphi^{-t}_H(x, P + \nabla_x v_-(P,x))) \quad \text{for } t \geq 0 \tag{5.177}$$

as we have choosen in (5.168). The same arguments for negative times lead to

$$\mathbf{g}_-(t,P,x) = g_-(P, \pi\circ\varphi^{-t}_H(x, P + \nabla_x v_+(P,x))) \quad \text{for } t \leq 0. \tag{5.178}$$

as we have choosen in (5.169). □

Remark 5.4. *Let* $\psi_\hbar^\pm(s,x) := e^{-\frac{i}{\hbar}\mathrm{Op}_\hbar^w(H)s}\varphi_\hbar^\pm(x)$*, define the position density* $\rho_\hbar^\pm(s,x) := |\psi_\hbar^\pm(s,x)|^2$ *and the (formal) current density* $J_\hbar^\pm(s,x) := \hbar\,\mathrm{Im}((\psi_\hbar^\pm)^\star\nabla_x\psi_\hbar^\pm(s,x))$*. The (formal) conservation law reads*

$$\partial_t\rho_\hbar^\pm(t,x) + \mathrm{div}_x J_\hbar^\pm(t,x) = 0. \tag{5.179}$$

In the next result we exhibit the well-posed setting.

Proposition 5.5. *Let* $\psi_\hbar^\pm(s,x) := e^{-\frac{i}{\hbar}\mathrm{Op}_\hbar^w(H)s}\varphi_\hbar^\pm(x)$*,* $\rho_\hbar^\pm(s,x) := |\psi_\hbar^\pm(s,x)|^2$*. Let* $\varphi_{\hbar,\varepsilon}^\pm \in C^\infty(\mathbb{T}^n;\mathbb{C})$ *such that* $\|\varphi_{\hbar,\varepsilon}^\pm - \varphi_\hbar^\pm\|_{H^1} \to 0$ *as* $\varepsilon \to 0^+$*. Define* $J_{\hbar,\varepsilon}^\pm(s,x) := \hbar\,\mathrm{Im}((\psi_{\hbar,\varepsilon}^\pm)^\star\nabla_x\psi_{\hbar,\varepsilon}^\pm(s,x))$ *and take a distributional limit* $J_\hbar^\pm := \lim_{\varepsilon\to0^+} J_{\hbar,\varepsilon}^\pm$ *in* $\mathcal{D}'([0,T]\times\mathbb{T}^n)$*. Then,*

$$\int_0^t\int_{\mathbb{T}^n}\partial_s f(s,x)\rho_\hbar^\pm(s,x) + \nabla_x f(s,x)\cdot J_\hbar^\pm(s,x)\;dxds = 0 \quad \forall f\in C^\infty([0,t]\times\mathbb{T}^n;\mathbb{R}). \tag{5.180}$$

cont-w

Proof. This equation well posed. Indeed,

$$\begin{aligned}
E[\psi_{\hbar,\varepsilon}^\pm(s)] &:= \int_{\mathbb{T}^n}\frac{\hbar^2}{2}|\nabla_x\psi_{\hbar,\varepsilon}^\pm(s,x)|^2 + V(x)|\psi_{\hbar,\varepsilon}(s,x)|^2dx && (5.181)\\
&= \int_{\mathbb{T}^n}\frac{\hbar^2}{2}|\nabla_x\psi_{\hbar,\varepsilon}^\pm(0,x)|^2 + V(x)|\psi_{\hbar,0}(s,x)|^2dx && (5.182)\\
&\to \int_{\mathbb{T}^n}\frac{\hbar^2}{2}|\nabla_x\varphi_\hbar^\pm(x)|^2 + V(x)|\varphi_\hbar^\pm(x)|^2dx \quad \text{as} \quad \varepsilon\to0^+ && (5.183)\\
&= \int_{\mathbb{T}^n}\Big(\frac{1}{2}|P+\nabla_x v_\pm(P,x)|^2 + V(x)\Big)|a_{\hbar,P}^\pm(x)|^2dx + \int_{\mathbb{T}^n}\frac{\hbar^2}{2}|\nabla_x a_{\hbar,P}^\pm(x)|^2dx\\
&= \bar{H}(P) + \int_{\mathbb{T}^n}\frac{\hbar^2}{2}|\nabla_x a_{\hbar,P}^\pm(x)|^2dx < +\infty \quad \forall\ 0<\hbar<1 && (5.184)
\end{aligned}$$

since $\hbar\|\nabla_x a_{\hbar,P}^\pm\|_{L^2}\to0$ thanks to the setting of $a_{\hbar,P}^\pm$.
Hence $\|J_{\hbar,\varepsilon}^\pm(s,\cdot)\|_{L^1} \le \|\psi_{\hbar,\varepsilon}(s,\cdot)\|_{L^2}\|\hbar\nabla_x\psi_{\hbar,\varepsilon}(s,\cdot)\|_{L^2} \le c\,\|\hbar\nabla_x\psi_{\hbar,\varepsilon}^\pm(s,\cdot)\|_{L^2} < +\infty$ uniformly in $(\varepsilon,s)\in(0,1]\times[0,t]$. We can take a distributional limit $J_\hbar^\pm := \lim_{\varepsilon\to0^+}J_{\hbar,\varepsilon}^\pm$ in $\mathcal{D}'([0,T]\times\mathbb{T}^n)$ and this gives

$$\lim_{\varepsilon\to0^+}\int_{\mathbb{T}^n}\nabla_x f(s,x)\cdot J_{\hbar,\varepsilon}^\pm(s,x)\;dx = \int_{\mathbb{T}^n}\nabla_x f(s,x)\cdot J_\hbar^\pm(s,x)\;dx \quad \forall s\in[0,t]$$

Since $\rho_{\hbar,\varepsilon}^\pm$ is weak-$\star$ converging to the unique $\rho_\hbar^\pm\in L^1([0,T]\times\mathbb{T}^n;\mathbb{R}^+)$, we deduce that the equation (5.180) solved by $(\rho_{\hbar,\varepsilon}^\pm(s,x), J_{\hbar,\varepsilon}^\pm(s,x))$ (in the distributional and in the strong sense) is also fulfilled by $(\rho_\hbar^\pm(s,x), J_\hbar^\pm(s,x))$ in the distributional sense. □

The last result of the section reads

Corollary 5.6. *Fix* $P\in\mathbb{R}^n$*, suppose that* $v_+(P,\cdot) = v_-(P,\cdot)\in C^2(\mathbb{T}^n;\mathbb{R})$ *and* $g(P,\cdot)\in W^{1,\infty}(\mathbb{T}^n;\mathbb{R}^+)$*. Then,* $\mathbf{g}_\pm$ *as in Theorem 5.1 fulfill* $\mathbf{g}_+ = \mathbf{g}_-$*,* $\mathbf{g}_\pm\in L^1([0,T];W^{1,\infty}(\mathbb{T}^n;\mathbb{R}^+))$ *and solves the transport equation*

$$\partial_t\mathbf{g}_\pm(t,P,x) + (P+\nabla_x v_\pm(P,x))\cdot\nabla_x\mathbf{g}_\pm(t,P,x) = 0 \quad \text{for} \quad t\in\mathbb{R} \tag{5.185}$$

tr11

with inital datum $\mathbf{g}_\pm(0,P,x) := g(P,x)$*.*

Proof. The regularity $v_\pm(P,\cdot)\in C^2(\mathbb{T}^n;\mathbb{R})$ implies the C^1-regularity of the vector field $P+\nabla_x v_\pm(P,\cdot)$ on $\mathbb{T}^n$. By standard transport PDE arguments (see for example [1]) we get the above equations. □

6 Appendix

equi-op **Lemma 6.1.** *Let* $\hat{H}_\hbar := -\frac{1}{2}\hbar^2\Delta_x + V(x)$*,* $H := \frac{1}{2}|\eta|^2 + V(x)$ *and* $\mathrm{Op}^w_\hbar(H)$ *as in (2.10). Then,*

$$\mathrm{Op}^w_\hbar(H)\psi = \hat{H}_\hbar\psi, \quad \forall\, \psi \in C^\infty(\mathbb{T}^n;\mathbb{C}). \tag{6.186}$$ equi-H

Proof. To begin, we recall that

$$\mathrm{Op}^w_\hbar(b)\psi(x) = (b(X, \frac{\hbar}{2}D) \circ T_x\, \psi)(x). \tag{6.187}$$ eq-02

where $(T_x\psi)(y) := \psi(2y - x)$, see Section 2.1. Morever, it is easily proved that when $H = \frac{1}{2}|\eta|^2 + V(x)$

$$H\Big(X, \frac{\hbar}{2}D\Big)\psi = \hat{H}_\hbar\psi \tag{6.188}$$ eq-02

for $\psi \in C^\infty(\mathbb{T}^n;\mathbb{C})$ and that $(\hat{H}_\hbar T_x\psi)(x) = \hat{H}_\hbar\psi(x)$. Thus, by (2.11) and (6.188) we get the statement.
□

L-reg **Remark 6.2.** *The operator* $\hat{H}_\hbar : H^2(\mathbb{T}^n;\mathbb{C}) \to L^2(\mathbb{T}^n;\mathbb{C})$ *is linear, selfadjoint and continuous. Hence, by standard results of evolution equations in Banach spaces, the solution of the Schrödinger equation (3.69) fulfills* $\psi_\hbar \in C^0(\mathbb{R}; H^2(\mathbb{T}^n;\mathbb{C})) \cap C^1(\mathbb{R}; L^2(\mathbb{T}^n;\mathbb{C}))$*. The one parameter group of unitary operators* $e^{-\frac{i}{\hbar}\mathrm{Op}^w_\hbar(H)t}$ *can be defined on* $L^2(\mathbb{T}^n;\mathbb{C})$ *and* $e^{-\frac{i}{\hbar}\mathrm{Op}^w_\hbar(H)t}\varphi \in C^0(\mathbb{R}; L^2(\mathbb{T}^n;\mathbb{C}))$ *(see for example [24]).*

Remark 6.3. *We recall that* $\mathbf{b} \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$ *consist of* $\mathbf{b} \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n;\mathbb{R})$ *satisfying (2.8). The Weyl quantization on* $\mathbb{R}^n$ *of these simbols reads*

$$\mathbf{b}^w(X,\hbar D)\psi(x) := (2\pi\hbar)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{\hbar}\langle x-y,\eta\rangle}\, b\Big(\frac{x+y}{2}, \eta\Big)\psi(y)dyd\eta, \quad \psi \in \mathcal{S}(\mathbb{R}^n;\mathbb{C}). \tag{6.189}$$ W-Rn

Notice that in the case of $H(x,\eta) = \frac{1}{2}|\eta|^2 + V(x)$ *with* $V \in C^\infty(\mathbb{R}^n;\mathbb{R})$*, it holds the equivalence* $H^w(X,\hbar D) = \hat{H}_\hbar$ *on the domain* $\mathcal{S}(\mathbb{R}^n;\mathbb{C})$ *(see for example Section 2.7 in [21]). By the identification* $\mathbb{T}^n \equiv (\mathbb{R}/2\pi\mathbb{Z})^n$*, if* V *is* $2\pi\mathbb{Z}^n$*-periodic then we could restrict* $H^w(X,\hbar D) : C^\infty(\mathbb{T}^n;\mathbb{C}) \to C^\infty(\mathbb{T}^n;\mathbb{C})$*. Obviously, this restriction cannot be done for all simbols in* $S^m(\mathbb{R}^n \times \mathbb{R}^n)$ *which are* $2\pi\mathbb{Z}^n$*-periodic in x-variables. For a more detailed and general discussion about the link between Pseudodifferential Operators on* $\mathbb{T}^n$ *and Pseudodifferential Operators on* $\mathbb{R}^n$ *which are* $2\pi\mathbb{Z}^n$*-periodic in x-variables, we address the reader to Section 6 in [25].*

The following result is shown in [27].

LemmaYang **Lemma 6.4.** *Let* X *be a metric space. Let* $d\mu_j$ $j \in \mathbb{N}$ *and* $d\mu$ *Borel probability measures on* X *such that* $d\mu_j \overset{\mathrm{w}-\star}{\longrightarrow} d\mu$ *as* $j \to +\infty$*. Let* $f_k, f\colon X \longrightarrow \mathbb{R}$ *(*$k \in \mathbb{N}$*) be Borel measurable functions such that*

$$\lim_{\lambda\to+\infty}\ \sup_{k\in\mathbb{N}} \int_{\{x\in X;\ |f_k(x)|>\lambda\}} |f_k(x)|\ d\mu_k(x) = 0. \tag{6.190}$$ eqYang1

Let

$$E := \Big\{x \in X;\ \exists\{x_k\}_{k\in\mathbb{N}} \subset X,\ \lim_{k\to+\infty} x_k = x,\ \lim_{k\to+\infty} f_k(x_k) \neq f(x)\Big\}. \tag{6.191}$$ eqYang2

If $\mu(E) = 0$ *then*

$$\lim_{j\to+\infty}\int_X f_j(x)d\mu_j(x) = \int_X f(x)d\mu(x).$$

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